Finite Capacity Single Server Queue with Flexible Service Rates.

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Abstract

The model discussed in this article has the assumption arrival process is poisson with state-dependent parameters, service time distribution is negative exponential with state-dependent rate, if there are n customers in the system then the arrival rate λ_n , Service rate μ_n, when an arrival finds the server free, the customer immediately enters the service station, otherwise the customer enters into a queue of size N(N>0), in the waiting line the waiting customer applies First come First service discipline and during the service period of the customer, if the customer satisfied with the service at any instant of the service period, the customer leaves the system with probability δ or continue with probability 1- δ, now the service rate becomes $\frac{1}{\delta}$ (service rate) (0< $\delta \leq 1$). This model defined using the infinitesimal generator matrix and for the analysis, the group generalised inverse of the infinitesimal generator is used. Using the group generalised inverses, the steady-state probabilities are obtained analytically. Performance measures based on the same are derived. In addition, some numerical illustrations are provided.

Keywords: Finite capacity Markovian queueing system, State-dependent rates, Flexible services, Infinitesimal generator matrix, Group generalised inverse, Steady state probabilities, and Performance measures.

1. Introduction

The theory of queues is an important branch of advanced probability theory, using queues real-life situations involving congestions can be effectively modelled. In a queue, the queue length and waiting time in the queue are the main factors affecting the patiency of the arriving or waiting customers. So, some remedial measures are to be taken to overcome the problem. One such solution is to modify the service rate based on the needs of the customer. In an M/M/1 queueing system, the arrival and service rates are constants. But in many practical situations, the arrival and service rates are state-dependent. That is, the assumption of independence and constant arrival and service rates can be relaxed to the extent of making each of the rates as λ_n and μ_n respectively, where n is the number of customers in the system (Cox and Smith(1961)). Some exceptional cases that received considerable attention based on physical scenarios in the literature are:

$$\lambda_n = \begin{cases} n\lambda, & n > 0 \\ \dfrac{\lambda}{n+1}, & n \geq 0 \end{cases}$$

$$\mu_n = \begin{cases} n\mu, & n \geq 1 \\ \mu, & n \geq 1 \end{cases}$$

Notable references are Conway and Maxwell (1961), Harris (1967), Scott (1970), Natvig (1973),Hadidi (1974), Conolly (1975), Boxmaet.al (2005), Kalyanaraman and Pattabiraman (2011) and Kalyanaraman and Sundaramoorthy (2019). Jonckheere and Borst (2006) analysed queues with state-dependent service rates. Narayanan and Manoharan (2007) considered non-linear arrival and service rates. Raheed and Manoharan (2014) modified the service rate by introducing the concept of service switches. In the queueing literature, most of the works are related to systems with infinite waiting lines, however, in many real-time queueing situations, the waiting line capacity is finite. For a finite capacity model, the waiting space is finite, the arriving customers leave the system if the line is full, such a model is called the loss model, and was first investigated by Erlang (1917). Some more works in this line are Fortet (1948), Vaulot (1954), Takacs(1969), Jagerman (1974), Harel (1987), Berezner et al.,(1995) and Kalyanaraman and Pattabiraman (2010).

In this paper, we analysed a finite-capacity Markovian queueing system with state-dependent arrival and service rates and flexible service rates. This model is defined using an infinitesimal generator matrix. The model defined above has been analysed using the method of group generalized inverses. Hunter (1969) concluded that a square matrix G processes the group inverse if the rank of G is equal to the rank of G^2. Some notable works in this area are Adi-Ben Israel and Greville (1974), Boullion and Odell (1971) and Campbell and Meyer (1979). Meyer (1975) provided a formula for a group inverse of an infinitesimal generator of m-state ergodic processes. Utilizing the group inverse he got the fixed probability vector. Hunter (1969) and Kemney& Snell (1960), both works obtained the mean first passage time matrix for m-state ergodic processes.

This paper is organised as follows: In sections 2 and 3, the method and the corresponding analysis are given. In section 4, the model definition is given. In section 5, the mathematical definition of the queueing system and the analysis are given. In section 6, some performance measures are given. In section 7, some numerical illustrations are provided. Finally, in section 8, a conclusion is given.

2. The Method

Let (X_n, Y_n), $n \geq 0$ be a Markov process on the state $S = \{(n,j): 0 \leq n \leq N,\ 1 \leq j \leq a_n\}$ with the following bock tridiagonal infinitesimal generator.

$$Q=\begin{pmatrix} B_0 & A_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ C_1 & B_1 & A_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & C_2 & B_2 & A_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & C_{N-1} & B_{N-1} & A_{N-1} \\ 0 & 0 & 0 & \cdots & \cdots & 0 & C_N & B_N \end{pmatrix}$$

Where $B_0, B_1, \dots, B_N$ are square matrices of order $a_0, a_1, \dots, a_N$ respectively. Their diagonal elements are strictly negative, the other elements are non-negative. The matrices $A_0, A_1, \dots, A_{N-1}$,$C_1, C_2, \dots, C_N$ are rectangular matrices and non-negative. The row sums of Q are equal to 0. That is,

$$B_0 e + A_0 e = 0$$
$$C_i e + B_i e + A_i e = 0: 1 \le i \le N-1,$$
$$C_N e + B_N e = 0$$

Where e denotes the column vector and unit elements.

For the determination of the stationary probability distribution, the following realization of the Markov chain is useful. Observe the process Q during the interval of time spent at the level$n$, before the original process enters the level $n+1$ for the first time. DenoteP_n, be the realization of the process. The state space of P_n is $S_n = \{(n,j): 1 \le j \le a_n\}$. All P_n, $0 \le n \le N-1$ are transient Markov Chains. The process p_N is the realization of process Q with state space $S_N = \{(N,j): 1 \le j \le a_N\}$, it is an ergodic Markov chain. Denote Q_n as the infinitesimal generator of the process P_n, $0 \le n \le N$. Let $W = (w_0, w_1, w_2, \dots, w_N)$ be the probability vector, where W_n be the probability that there are n customers in the system.

3. The Algorithm

Based on the method described in the above sub-section the following algorithm is proposed to solve the model defined in this paper:

Step 1 Write $R=\begin{pmatrix} U & c \\ d' & \alpha \end{pmatrix}$ $R=-Q$, U is $(m-1)\times(m-1)$ matrix

Step 2: Check the rank of R = rank of R^2, if it is true $R^{\#}$ exists.

Step 3: Calculate $h' = d'U^{-1}$ and $\beta = 1 - h'j$ where $\beta$is non-zero.

Step 4: Calculate $W' = \frac{1}{\beta}[-h', 1]$.

4. The Model

The model has the following characteristics:

I. The queueing model has a single server with a finite waiting line of size N.

II. The arrival process is a Poisson process with state-dependent parameters.
III. Service time distribution is negatively exponential with state state-dependent rate.
IV. If there are n customers in the system then, the arrival rate λ_n and service rate μ_n

V. When an arrival finds the service free, the customer immediately enters the service station, otherwise, the customer enters into a queue of size N (N>0)
VI. In the waiting line, the waiting customer applies the First come First service discipline.
VII. During the service period of the customer, if the customer is satisfied with the service at any instant of the service period, the customer leaves the system with probability δ or continues with probability 1-δ, the service rate becomes $\frac{1}{\delta}$ (service rate) $(0 < \delta \leq 1)$

The Markov chain related to the model defined above is $\{(X_n, Y_n): n \geq 0\}$ with state space.$S = \{(i,j): 0 \leq i \leq N,\ j = a_0, a_1, a_2\}$.

The Q matrix is.

$$Q = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{\mu_1}{\delta} & -\left(\lambda_1 + \frac{\mu_1}{\delta}\right) & \lambda_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{\mu_2}{\delta} & -\left(\lambda_2 + \frac{\mu_2}{\delta}\right) & \lambda_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \frac{\mu_{N-1}}{\delta} & -\left(\lambda_{N-1} + \frac{\mu_{N-1}}{\delta}\right) & \lambda_{N-1} \\ 0 & 0 & 0 & \cdots & \cdots & 0 & \frac{\mu_N}{\delta} & -\frac{\mu_N}{\delta} \end{pmatrix}$$

Where $a_0 = \{0\},\ a_1 = \{1,2,3, \dots,\ N-1\},\ a_2 = \{N\}$.

Define $R = \begin{pmatrix} U & c \\ d' & \alpha \end{pmatrix}$, where $R = -Q$, U is an $N \times N$ matrix, corresponding to the states $\{0,1,2, \dots, N\}$.

5. The Mathematical Model and Analysis

The model defined in the article can be identified using the above construction. For the solution, we use the technique of group inverse (Kalyanaraman & Pattabiraman (2010)). The Markov chain related to the model discussed in this article is $\{(X_n, Y_n): n \geq 0\}$ with state space.$S = \{(i,j): 0 \leq i \leq N,\ j = a_0, a_1, a_2\}$.

To apply the algorithm of section 4, we take $R = \begin{pmatrix} U & c \\ d' & \alpha \end{pmatrix}$ where

$R = -Q$,

$$U=\begin{pmatrix} \lambda_0 & -\lambda_0 & 0 & 0 & \cdots & 0 \\ -\frac{\mu_1}{\delta} & \lambda_1+\frac{\mu_1}{\delta} & -\lambda_1 & 0 & \cdots & 0 \\ 0 & -\frac{\mu_2}{\delta} & \lambda_2+\frac{\mu_2}{\delta} & -\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{\mu_{N-1}}{\delta} & \lambda_{N-1}+\frac{\mu_{N-1}}{\delta} \end{pmatrix}$$

$$\mathbf{c}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ -\lambda_{N-1} \end{pmatrix}, \quad d'=\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -\frac{\mu_N}{\delta} \end{pmatrix} \text{ and } \alpha=\left(\mu_N\right)$$

We analyse the above model using the method described in Section 4,

U^{-1} is given by, $U^{-1}=(u_{ij})_{(N-1)\times(N-1)}$

Where,

$$u_{ij}=\frac{1}{\lambda_{j-1}}\left(1+\sum_{k=1}^{N-2}\prod_{l=j}^{k}\frac{\mu_l}{\delta\lambda_l}\right), \quad \text{for } i\le j$$

$$u_{ij}=\frac{1}{\lambda_{j-1}}\sum_{k=i-j}^{N-2}\prod_{l=j}^{k}\frac{\mu_l}{\delta\lambda_l}, \quad \text{for } i>j$$

The unique fixed probability vector $W'=\frac{1}{\beta}(W_0,W_1,\ldots,W_N)$

$$\beta=\frac{1}{\prod_{i=0}^{N-1}\lambda_i}\left[\prod_{i=1}^{N}\frac{\mu_i}{\delta}+\sum_{k=1}^{N-1}\left(\prod_{i=0}^{k-1}\lambda_i\prod_{j=k+1}^{N}\frac{\mu_j}{\delta}\right)+\prod_{i=0}^{N-1}\lambda_i\right]$$

Where,

$$W_0=\frac{1}{\delta K}\prod_{j=1}^{N}\mu_j$$

$$W_i=\frac{1}{K}\left(\prod_{j=0}^{i-1}\lambda_j\prod_{j=i+1}^{N}\frac{\mu_j}{\delta}\right)$$

$$W_N = \frac{1}{K}\prod_{j=0}^{N-1}\lambda_j$$

Where, $$K = \prod_{i=1}^{N}\frac{\mu_i}{\delta} + \sum_{k=1}^{N-1}\left(\prod_{i=0}^{k-1}\lambda_i \prod_{j=k+1}^{N}\frac{\mu_j}{\delta}\right) + \prod_{i=0}^{N-1}\lambda_i$$

6. Some Performance measures.

In this section, we present the following performance measures related to the model discussed in this paper:

I. The Mean number of customers in the system, $L = \sum_{n=1}^{N} nW_n$

II. The probability that the server is idle, W_0

III. The Blocking probability, $P_\beta = 1 - W_N$

7. Numerical Analysis

In this segment, we examine three specifics (denoted by Mode I, Model II, and Model III) related to the queueing system discussed in this article. In this study, we take $\lambda_n = \frac{1}{n+1}, \mu_n = \frac{1}{2n}$ and vary the values of $N = 5$(Model I) and $N = 7$(Model II)

Table 7.1 Model I: $M^{(n)}/M^{(n)}/1/5$

Parameter	μ_1	μ_2	μ_3	μ_4	μ_5
Value	**0.5**	**0.25**	**0.1666**	**0.125**	**0.1**
Parameter	λ_0	λ_1	λ_2	λ_3	λ_4
Value	**1**	**0.5**	**0.333**	**0.25**	**0.2**
Parameter	δ				
Value	**0.5**				

$$R = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1.5 & -0.5 & 0 & 0 & 0 \\ 0 & -0.5 & 0.833 & -0.333 & 0 & 0 \\ 0 & 0 & -0.334 & 0.584 & -0.25 & 0 \\ 0 & 0 & 0 & -0.25 & 0.45 & -0.2 \\ 0 & 0 & 0 & 0 & -0.10 & 0.10 \end{pmatrix}$$

$$U^{-1}=\begin{pmatrix}15.030 & 14.03 & 11.903 & 8.999 & 4.999\\ 14.030 & 14.03 & 11.903 & 8.999 & 4.999\\ 12.030 & 12.03 & 11.903 & 8.999 & 4.999\\ 12.030 & 8.9027 & 8.9027 & 8.999 & 4.999\\ 12.030 & 4.9015 & 4.9015 & 4.999 & 4.999\end{pmatrix}$$

Rank of Q=Rank of Q^2=5

$$d'=\begin{pmatrix}0 & 0 & 0 & 0 & -0.10\end{pmatrix}$$

$$h'=\begin{pmatrix}-1.2030 & -0.50150 & -0.50150 & -0.50 & -0.50\end{pmatrix}$$

$$\beta=4.205$$

$$W'=\begin{pmatrix}0.2855 & 0.1190 & 0.1190 & 0.1190 & 0.1190 & 0.2399\end{pmatrix}$$

L=2.38, W_0=0.286 and $P_\beta=0.762$

Table 7.2 Model II: $M^{(n)}/M^{(n)}/1/7$

Parameter	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7
Value	**0.5**	**0.25**	**0.167**	**0.125**	**0.1**	**0.083**	**0.071**
Parameter	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
Value	**1**	**0.5**	**0.333**	**0.25**	**0.2**	**0.167**	**0.143**
Parameter	δ						
Value	**0.5**						

$$R=\begin{pmatrix}1 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\ -1 & 1.5 & -0.5 & 0 & 0 & 0 & 0 & 0\\ 0 & -0.5 & 0.833 & -0.333 & 0 & 0 & 0 & 0\\ 0 & 0 & -0.334 & 0.584 & -0.25 & 0 & 0 & 0\\ 0 & 0 & 0 & -0.25 & 0.45 & -0.2 & 0 & 0\\ 0 & 0 & 0 & 0 & -0.2 & 0.367 & -0.167 & 0\\ 0 & 0 & 0 & 0 & 0 & -0.166 & 0.309 & -0.143\\ 0 & 0 & 0 & 0 & 0 & 0 & -0.142 & 0.142\end{pmatrix}$$

$$U^{-1}=\begin{pmatrix} 28.008 & 27.008 & 25.008 & 21.939 & 18 & 13.01 & 7 \\ 27.008 & 27.008 & 25.008 & 21.939 & 18 & 13.01 & 7 \\ 25.008 & 25.008 & 25.008 & 21.939 & 18 & 13.01 & 7 \\ 22.005 & 22.005 & 22.005 & 21.939 & 18 & 13.01 & 7 \\ 17.993 & 17.993 & 17.993 & 17.939 & 18 & 13.01 & 7 \\ 13.01 & 13.01 & 13.01 & 13.01 & 13.01 & 13.01 & 7 \\ 7 & 7 & 7 & 7 & 7 & 7 & 7 \end{pmatrix}$$

Rank of Q=Rank of Q^2=7

$$d'=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -0.142 \end{pmatrix}$$

$$h'=\begin{pmatrix} -0.99 & -0.99 & -0.99 & -0.987 & -0.987 & -0.987 & -0.993 \end{pmatrix}$$

$$\beta = 7.924$$

$$W'=\begin{pmatrix} 0.124 & 0.124 & 0.124 & 0.126 & 0.126 & 0.126 & 0.125 & 0.126 \end{pmatrix}$$

L=4.356, W_0=0.124 and $P_\beta = 0.874$

8. Conclusion

A finite capacity single server Markovian queue with state-dependent rate has been considered in this paper. In addition, the service rate is modified using Bernoulli probability. The model, we analysed using group generalized inverses, also obtain some performance measures. We also provide some numerical illustrations.

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