Steady state analysis for a single server finite source, finite waiting line queue with accessible batch service and with statistical process control for number of customers

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Abstract

This paper analyzes a finite source single server Poisson arrival process, negative exponentially distributed service time and queue with state dependent parameter queue. The source size is N and if the arriving customer finds the server is busy, the customer waits in a queue of size $L(< N)$. The services are given in batches of size $j, 1 \leq j \leq M(< N)$. At the time of arrival, if the number of customers in the service station is less than M, then the arrival joins the service batch and if the number of customers in the service station is M, then the new arrival joins a waiting line. After completing service, the customers leave from the service station singly instead of batches. Using Markov process and recursive technique, we derive the stationary system length distributions at arbitrary epoch. Various performance measures are presented. Statistical process control for mean number of customers is carried out using control chart analysis. Finally, some numerical results showing the effect of model parameters on key performance measures of the system are presented.

Keywords: Finite Source Queue-Single Departure-Batch Service-Accessible Batch-Transient Analysis-Steady State Analysis-Performance Measures-Control chart analysis.
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1 Introduction

A finite source queueing model is a type of queueing system where the potential number of customers(or sources of requests) is limited to a fixed number. This contrasts with infinite source model, where the arrival of new customers is assumed to be unlimited. Finite source models are commonly used in situations where the customer population is constrained, such as repair systems, healthcare facilities, or industrial maintenance scenarios. In this system the number of potential customers(or sources) is fixed at N. If a customer enters the system for service, the population of available customers decreases. The arrival rate is dependent on the number of customers not currently in the system since only those outside can generate new requests. If n customers are already in the system, the arrival rate is proportional to $N-n$. Some real life related situations are Repair and Maintenance Systems: Limited machines needing service, Healthcare: Hospital beds or staff serve a finite patient population, Inventory Systems: Restocking from limited warehouses and Call Centers: Small teams serve a limited customer base.

In a bulk service queue the server manages large volumes of tasks, requests, or operations in an organized manner. This can apply to various contexts, such as customer service, IT operations, logistics, or any environment where multiple simultaneous tasks need to be processed efficiently. A queue for bulk processing of data, updates, or tasks, such as deploying software updates across multiple devices and scheduling and handling bulk shipments in warehouses or delivery systems are some practical applications of bulk service queues. The foremost work related to queue with single server bulk services by Bailey(1954)(also Downton 1955). The authors considered that customers are served in batches of not more than b. If, the server finds more than b customers waiting for service, at a service completion point he takes a batch of b customers for service while the others will wait. On the other hand, if he finds $r(0 \leq r \leq b)$ customers, he takes all the r customers as a batch for service. Neuts(1967) considered the same rule with the restriction that $(1 \leq r \leq b)$, called bulk service rule. The two notable works in the earlier stage are by Bloemena(1960), Jaiswal(1961). Fabens(1961) and Tackacs(1962) considered a service rule in which services are given batches, is a random variable Y. Some more notable works are by Medhi and Borthakur(1972), Medhi(1975, 1979), Chaudhry and Templeton(1983), Chaudhry etal(1984), Briere and Chaudhry(1988) and Chaudhry and Gupta(1992). Markovian systems with accessible batches for service have been studied by Sivasamy(1990). In this rule, the services are given in batches of fixed size, say K, and whereas at the beginning of a service, if there are less than K customers in the system, the server starts service to all the customers present and arrival are admitted to the serving batch, until the batch size becomes K. This rule is, usually used in transportation systems.

In a queueing model, if there is no waiting space or a finite waiting space, then such a queueing model is called loss model. In other words, a loss queueing model is a type of queueing system where arriving customers that find the system fully occupied are not allowed to wait in a queue. Instead, they are immediately rejected or "lost". These models are often used to represent systems with no waiting space, such as telephone networks, circuit-switched systems, or call centers where excess traffic is dropped or system typically has a finite number of servers or channels, as in the case of transportation(Parking Systems with a fixed number of spaces), Healthcare(Emergency departments with limited beds) and Manufacturing(Systems with limited processing stations and no buffer space). In all the above situations, once all are occupied, new arrivals are blocked.

The need for statistical process control(SPC) arises if variability occurs in manufacturing processes and also it is true that no two manufactured items are exactly alike. When the random causes are alone present then we say that the process is "in control", on the other hand, when assignable causes are present, the process is out of control. During the process of production, the lots are sending for the quality control unit and the items are queued for their turn. After examination the items leave the system. In the process of production one of important factor is to sustain the quality. It is done through controlling, improving and maintaining the quality of the product. Theoretically, it is done through statistical quality control methods. The statistical quality control comprises design of experiment(Montgomery, 2012) statistical process control and acceptance sampling plan. Control chart is a statistical technique applied to control deviations of any repetitive process. The cntrol chart contains central line (CL), which shows the desired standard, upper control line (UCL), which shows the upper limit for the tolerance of desired standard and lower control line (LCL), which shows the lower limit for the tolerance of desired standard of the quality characteristics to be observed for the process. Now $CL = \mu$, $UCL = \mu + L\sigma$, $LCL = \mu - L\sigma$ where L is the distance of central limits from the CL expressed in standard units. The general theory of control chart was first proposed by Shewhart, the corresponding control chart is called Shewhart control chart. In the chart, the X axis represents the sample points and Y axis represents the quality character. The distribution of the plotted statistic is approximated by a normal distribution, with parameters mean and standard deviation is the basic principle of Shwehart chart. This chart is called C_1 chart. The parameters of the C_1 chart are $CL = \mu$, $UCL = \mu + 3\sigma$, $LCL = \mu - 3\sigma$, where μ and σ are the sample mean and standard deviation of the quality characters studied. Many researchers contributed towards queueing theory. But a few works appears in the area of combination of queueing theory and quality control. Shore(2000) constructed control chart for $M/M/S$ queue. Shore(2006) developed Shewhart like general control charts for $G/G/S$ queueing system. Khaparade and Dhabe(2010) obtained the control chart for the queue length of $M/M/1$ system. Kalyanaraman and Shakila(2022) calculated the performance of the systme using control chart analysis of a batch arrival heterogeneous two server queue with breakdown, restricted admissibiliity, discouraged arrivals.

In this paper, we consider a finite source single server queue with Poisson arrival process, negative exponentially distributed service time. The service parameter is system state dependent. The source size is N and if the arriving customer finds the server is busy, the customer waits in a queue of size $L(< N)$. The services are given in batches of size $j, 1 \leq j \leq M(< N)$. At the time of arrival, if number of customers in the service station is less than M, then the arrival joins the service batch and if the number of customers in the service station is M, then the new arrival joins a waiting line. After completing service, the customers leave from the service singly instead of batches. The model definition and the analysis are given in section 2.

2 Model and Analysis

In this section, we introduce the mathematical definition, relevant notations and the analysis in transient state and steady state are given.

2.1 Model definition

The customers arrives from a source of size N. The arrival process follows Poisson with rate λ. Service times are random variables, follows exponential distributed with state dependent parameter i.e., depends on the number of customers undergoing service. The service rule is: The services are given in Batches of variable size $j(1 \leq j \leq M)$ and the maximum number of customers the service station can accommodate is $M(< N)$. In addition, the service batches are accessible batches. That is, at the time of arrival, if number of customers in the service station is less than M, then the arrival joins the service batch and if the number of customers in the service station is M, then the new arrival joins a waiting line of capacity $L(1 \leq L < N)$. If an arrival finds L customers in the waiting line, it doesn't join the waiting line (Loss to a system). In the waiting line, the first in first out (FIFO) queue discipline is used. The services are given in batches but the customers depart singly after completing service.

2.2 Notations

The following notations are introduced for the analysis: Let $X(t)$ be the number of cutomers in the Queue at time t and $Y(t)$ be the number of customers in the service station at time t. The two dimensional stochastic process $\{(X(t), Y(t)) : t \geq 0\}$ is a Markov Process with state space $S = \{0, 1, 2, ..., L\} \times \{0, 1, 2, ..., M\}$. Let $p(n, m, t) = Pr\{X(t) = n, Y(t) = m\}$ be the corresponding probability distribution and let $p(n, m) = \lim_{t\to\infty} p(n, m; t)$ be the corresponding steady state probability distribution.

2.3 The Transient Analysis

Using birth-death arguments the following differential-difference equations are obtained:

$$p'(0,0;t) = -N\lambda p(0,0;t) + \mu_1 p(0,1;t) \tag{2.1}$$

$$\begin{aligned} p'(0,n;t) = &-[(N-n)\lambda + n\mu_n]p(0,n;t) + (N-n+1)\lambda p(0,n-1;t) \\ &+(n+1)\mu_{n+1}p(0,n+1;t); \qquad 1 \le n \le M-1 \end{aligned} \tag{2.2}$$

$$\begin{aligned} p'(0,M;t) = &-[(N-M)\lambda + M\mu_M]p(0,M;t) \\ &+(N-M+1)\lambda p(0,M-1;t) + M\mu_M p(1,M;t) \end{aligned} \tag{2.3}$$

$$\begin{aligned} p'(n,M;t) = &-[(N-M-n)\lambda + M\mu_M]p(n,M;t) + (N-M-n+1) \\ &\lambda p(n-1,M;t) + M\mu_M p(n+1,M;t); \quad 1 \le n \le L-1 \end{aligned} \tag{2.4}$$

$$p'(L,M;t) = -M\mu_M p(L,M;t) + (N-M-(L-1))\lambda p(L-1,M;t) \tag{2.5}$$

The corresponding matrix form for equations (2.1) to (2.5) is

$$p'(t) = Ap(t) \tag{2.6}$$

where,

$$A = \begin{bmatrix} a_0 & \mu_1 & \dots & 0 & 0 & \dots & 0 & 0 \\ b_0 & a_1 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & b_1 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 \\ 0 & 0 & 0 & a_{M-1} & M\mu_M & \dots & 0 & 0 \\ 0 & 0 & 0 & b_{M-1} & a_M & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & M\mu_M & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & a_{M+L-1} & M\mu_M \\ 0 & 0 & 0 & 0 & 0 & \dots & b_{M+L-1} & a_{M+L} \end{bmatrix}$$

where,
$a_0 = -N\lambda$
$a_1 = -[(N-1)\lambda + \mu_1]$

$a_{M-1} = -[(N-M+1)\lambda + (M-1)\mu_{M-1})]$
$a_M = -[(N-M)\lambda + M\mu_M]$
$a_{M+L-1} = -[(N-M-L+1)\lambda + M\mu_M]$
$a_{M+L} = -M\mu_M$
$b_0 = N\lambda$
$b_1 = (N-1)\lambda$
$b_{M-1} = (N-M+1)\lambda$
$b_{M+L-1} = (N-M-L+1)\lambda$

$$p(t) = (p(0,0;t), p(0,1;t), ...p(0,M-1;t), p(0,M;t),$$

$$p(1,M;t), ...p(L-1,M;t), p(L,M;t))^T \tag{2.7}$$

Integrating the equation (2.6) and $p^{'}(t)$ is $\frac{d}{dt}p(t)$ we get,

$$\frac{p^{'}(t)}{p(t)} = A \tag{2.8}$$

$$p(t) = e^{At}.C \tag{2.9}$$

At $t = 0$, (2.9) becomes,

$$C = p(0) \tag{2.10}$$

Therefore, Equation(2.9) together with (2.10) gives the time dependent solution for the model and is,

$$p(t) = e^{At}.p(0) \tag{2.11}$$

where $p(0)$ is the initial probability vector.

For finding matrix exponential, Python provides sophisticated method powered by the SciPy library, we use the coding in Python, and find the values of e^{At} for various values of t and fixing the parameters $N = 20, L = 9, M = 10, \mu_i(i = 1,2,...,10) = 1, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9$. The probability vector p(t) is obtained using $p(t) = e^{At}$ .p(0), by taking the initial probability vector $p(0) = [1,0,...,0]^{'}$

2.4 The Steady State Analysis

In steady state, the following steady state equations are obtained from (2.1) to (2.5),

$$N\lambda p(0,0) = \mu_1 p(0,1) \tag{2.12}$$

$$\begin{aligned}[(N-n)\lambda + n\mu_n]p(0,n) = {} & (N-n+1)\lambda p(0,n-1) \\ & +(n+1)\mu_{n+1}p(0,n+1); \qquad 1 \leq n \leq M-1\end{aligned} \tag{2.13}$$

$$[(N-M)\lambda + M\mu_M]p(0,M) = (N-M+1)\lambda p(0,M-1) + M\mu_M p(1,M) \tag{2.14}$$

$$\begin{aligned}[(N-M-n)\lambda + M\mu_M]p(n,M) = {} & (N-M-n+1)\lambda p(n-1,M) \\ & +M\mu_M p(n+1,M); \quad 1 \leq n \leq L-1\end{aligned} \tag{2.15}$$

$$M\mu_M p(L,M) = (N-M-L+1)\lambda p(L-1,M) \tag{2.16}$$

and the normalization condition is,

$$p(0,0) + \sum_{m=1}^{M} p(0,m) + \sum_{m=1}^{L} p(m,M) = 1 \tag{2.17}$$

From (2.12) and (2.13), we get,

$$p(0,M-1) = \frac{N(N-1)(N-2)...(N-(M-2))\lambda^{M-1}}{(M-1)!\mu_1\mu_2...\mu_{M-1}} p(0,0) \tag{2.18}$$

$$p(0,M) = \frac{N(N-1)(N-2)...(N-(M-1))\lambda^{M}}{M!\mu_1\mu_2...\mu_{M}} p(0,0) \tag{2.19}$$

From (2.14),

$$p(1,M) = \frac{N(N-1)(N-2)...(N-M)\lambda^{M+1}}{M!(M\mu_M)\mu_1\mu_2...\mu_{M}} p(0,0) \tag{2.20}$$

From (2.15) and (2.16), we get,

$$p(L,M) = \frac{N(N-1)(N-2)...(N-(M+(L-1)))\lambda^{M+L}}{M!(M\mu_M)^L\mu_1\mu_2...\mu_{M}} p(0,0) \tag{2.21}$$

$$p(0,0)=\left\{1+\sum_{m=1}^{M}\frac{N(N-1)...(N-(m-1)))\lambda^{m}}{m!\mu_1\mu_2...\mu_m}\right.$$
$$\left.+\sum_{m=1}^{L}\frac{N(N-1)...(N-(M+(m-1)))\lambda^{M+m}}{M!(M\mu_M)^m\mu_1\mu_2...\mu_M}\right\}^{-1} \tag{2.22}$$

$$p(0,j)=\binom{N}{j}\prod_{i=1}^{j}\rho_i p(0,0), \qquad j=1,2,...,M \tag{2.23}$$

$$p(j,M)=\binom{N}{M+j}\prod_{i=1}^{M}\rho_i(\rho_{M+1})^j p(0,0), \qquad j=1,2,...,L \tag{2.24}$$

where,

$$\rho_i=\frac{\lambda}{\mu_i},\ \rho_{M+1}=\frac{\lambda}{M\mu_M}, \qquad i=1,2,...,M$$

On simplification, $p(0,0)$ becomes,

$$p(0,0)=\left\{1+\sum_{j=1}^{M}\binom{N}{j}\prod_{i=1}^{j}\rho_i+\sum_{j=1}^{L}\binom{N}{M+j}\prod_{i=1}^{M}\rho_i(\rho_{M+1})^j\right\}^{-1} \tag{2.25}$$

Equations (2.23), (2.24) together with equation (2.25) shows the steady state probabilities of the models discussed in this paper.

2.5 Some Performance Measures

In this section some performance measures like mean number of customers in the queue, in the system, in the service station and in the source, the idle probability are derived both in the case of time dependent domain(transient case) and time independent domain(stationary case) using statistical formulas.

2.5.1 Transient Case

1. Mean number of customers in the queue at time t

$$L_1(t)=\sum_{n=0}^{L}np(n,M;t) \tag{2.26}$$

2. Mean number of customers in the system at time t

$$L_2(t) = \sum_{n=0}^{M} np(0,n;t) + \sum_{n=1}^{L} np(n,M;t) \tag{2.27}$$

3. Mean number of customers in the service station at time t

$$L_3(t) = \sum_{n=0}^{M} np(0,n;t) + M\sum_{n=1}^{L} p(n,M;t) \tag{2.28}$$

4. Mean number of customers in the source at time t

$$L_4(t) = \sum_{n=0}^{M} (N-n)p(0,n;t) + \sum_{n=0}^{L} (L-n)p(n,M;t) \tag{2.29}$$

5. Idle Probability at time t

$$p(0,0;t) = \left\{ 1 + \sum_{j=1}^{M} \binom{N}{j} \prod_{i=1}^{j} \rho_i + \sum_{j=1}^{L} \binom{N}{M+j} \prod_{i=1}^{M} \rho_i (\rho_{M+1})^j \right\}^{-1} \tag{2.30}$$

2.5.2 Stationary Case

1. Mean number of customers in the queue

$$L_1 = \sum_{n=0}^{L} np(n,M) = \sum_{n=1}^{L} n\binom{N}{M+n} \prod_{i=1}^{M} \rho_i (\rho_{M+1})^n p(0,0) \tag{2.31}$$

2. Mean number of customers in the system

$$\begin{aligned} L_2 &= \sum_{n=0}^{M} np(0,n) + \sum_{n=1}^{L} np(n,M) \\ &= \sum_{n=1}^{M} n\binom{N}{n} \prod_{i=1}^{n} \rho_i p(0,0) + \sum_{n=1}^{L} n\binom{N}{M+n} \prod_{i=1}^{M} \rho_i (\rho_{M+1})^n p(0,0) \end{aligned} \tag{2.32}$$

3. Second moment of Mean number of customers in the system

$$\begin{aligned} L_3 &= \sum_{n=0}^{M} n^2 p(0,n) + \sum_{n=1}^{L} n^2 p(n,M) \\ &= \sum_{n=1}^{M} n^2\binom{N}{n} \prod_{i=1}^{n} \rho_i p(0,0) + \sum_{n=1}^{L} n^2\binom{N}{M+n} \prod_{i=1}^{M} \rho_i (\rho_{M+1})^n p(0,0) \end{aligned} \tag{2.33}$$

4. Mean number of customers in the service station

$$L_4 = \sum_{n=0}^{M} np(0,n) + M\sum_{n=1}^{L} p(n,M)$$
$$= \sum_{n=1}^{M} n\binom{N}{n}\prod_{i=1}^{n}\rho_i p(0,0) + M\sum_{n=1}^{L}\binom{N}{M+n}\prod_{i=1}^{M}\rho_i(\rho_{M+1})^n p(0,0) \tag{2.34}$$

5. Mean number of customers in the source

$$L_5 = \sum_{n=0}^{M}(N-n)p(0,n) + \sum_{n=0}^{L}(L-n)p(n,M)$$
$$= \sum_{n=0}^{M-1}(N-n)\binom{N}{n}\prod_{i=1}^{n}\rho_i p(0,0)$$
$$+ \sum_{n=0}^{L}(L-n)\binom{N}{M+n}\prod_{i=1}^{M}\rho_i(\rho_{M+1})^n p(0,0) \tag{2.35}$$

6. Idle Probability

$$p(0,0) = \left\{1 + \sum_{j=1}^{M}\binom{N}{j}\prod_{i=1}^{j}\rho_i + \sum_{j=1}^{L}\binom{N}{M+j}\prod_{i=1}^{M}\rho_i(\rho_{M+1})^j\right\}^{-1} \tag{2.36}$$

3 Waiting Time Analysis

Let W represents the time spent by an arriving customer(Test Customer) in the queue and W(t) be its cummulative distributive function. There are two cases (i) If the Test Customer finds no one in the system, its waiting time is the service time in the system. In this case $W = 0$. (ii) If the Test Customer finds the service station is full then the waiting time in the queue $W > 0$. Using simple probabilistic arguments the distribution of W is obtained as
(i) If $W = 0$,

$$W(0) = \Pr\{W = 0\}$$

$$W(0) = \Pr\{M-1 \text{ (or) less number of customers in the service station}\}$$

$$W(0) = \sum_{n=0}^{M-1}\binom{N}{n}\prod_{i=1}^{n}\rho_i p(0,0) \tag{3.1}$$

(ii) If $W > 0$,

$$W(t) = \Pr\{0 < W \leq t\}$$

$$W(t) = \sum_{n=M}^{L} p(n-M, M) \int_0^t e^{-M\mu_M x}.M\mu_M x.\frac{(M\mu_M)^{n-M}}{(n-M)!}dx \tag{3.2}$$

Now,

$$\int_0^t e^{-M\mu_M x}.M\mu_M x.\frac{(M\mu_M)^{n-M}}{(n-M)!}dx = 1 - \sum_{i=0}^{n-M} (M\mu_M t)^i.\frac{e^{-M\mu_M t}}{i!}$$

The cummulative distribution function for waiting time random variable W is

$$W(t) = \sum_{n=M}^{L} p(n-M, M)\left\{1 - \sum_{i=0}^{n-M} (M\mu_M t)^i.\frac{e^{-M\mu_M t}}{i!}\right\}$$

Now differentiating $W(t)$ with respect to t we get,

$$\frac{d}{dt}(W(t)) = \sum_{n=M}^{L} p(n-M, M)\left\{ - \sum_{i=0}^{n-M} \frac{(M\mu_M)^i}{i!}(t^i.(-M\mu_M)e^- M\mu_M t\right.$$

$$\left. +e^- M\mu_M t.it^{i-1})\right\}$$

$$= \sum_{n=M}^{L} p(n-M, M)\left\{ \sum_{i=0}^{n-M} \tfrac{(M\mu_M)^{i+1}.e^{-M\mu_M t}.t^i}{i!}\right.$$

$$\left. - \sum_{i=1}^{n-M} \tfrac{(M\mu_M)^i.e^{-M\mu_M t}.t^{i-1}}{(i-1)!}\right\}$$

3.1 Expected Mean Waiting Time

$$E(W) = \int_0^\infty t dW(t) \tag{3.3}$$

$$= \sum_{n=M}^{L} p(n-M, M)\left\{ \sum_{i=0}^{n-M} \tfrac{(M\mu_M)^{i+1}}{i!} \int_0^\infty e^{-M\mu_M t}.t^{i+1}dt\right.$$

$$\left. - \sum_{n=M}^{L} p(n-M, M) \sum_{i=1}^{n-M} \tfrac{(M\mu_M)^i}{(i-1)!} \int_0^\infty e^{-M\mu_M t}.t^i dt\right\}$$

Now, $\int_0^\infty e^{-M\mu_M t}.t^{i+1}dt = \frac{(i+1)!}{(M\mu_M)^{i+2}}$ and $\int_0^\infty e^{-M\mu_M t}.t^i dt = \frac{i!}{(M\mu_M)^{i+1}}$

Substituting above values in equation (3.2) we get,

$$E(W) = \sum_{n=M}^{L} p(n-M,M) \sum_{i=0}^{n-M} \frac{(M\mu_M)^{i+1}}{i!} \times \frac{(i+1)!}{(M\mu_M)^{i+2}}$$

$$- \sum_{n=M}^{L} p(n-M,M) \sum_{i=1}^{n-M} \frac{(M\mu_M)^{i}}{(i-1)!} \times \frac{i!}{(M\mu_M)^{i+1}}$$

$$E(W) = \sum_{n=M}^{L} p(n-M,M) \sum_{i=0}^{n-M} \frac{1}{M\mu_M}$$

$$E(W) = \sum_{n=M}^{L} p(n-M,M) \frac{(n-M+1)}{M\mu_M} \tag{3.4}$$

Equation (3.4) shows the expected mean waiting time of a customer.

4 The Numerical Study

In this section, we presents some numerical illustrations to show the effect of the parameters on the model, both transient case and steady state case in this section. By taking particular values to the parameters, λ, μ_i, M, L and N, the probabilities and performance measures are calculated and are presented in the following subsections.

4.1 Transient Case

For finding matrix exponential, Python provides sophisticated method powered by the SciPy library. We use the coding in the Python, and we find the value of e^{At} for various values of t and fixing the parameters $N = 20, L = 9, M = 10, \lambda = 5, \mu_i(i = 1, 2, ...10) = 1, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9$. The probability vector $p(t)$ is obtained using $p(t) = e^{At}$, by taking the initial probability vector $p(0) = [1, 0, 0, ...0]'$. The corresponding performance measures are calculated using the formulas in the subsection 2.5.1. The transient probabilities of various values of t are presented in table 4.1 and 4.2 and the performance measures are presented in table 4.3. The first row of table 4.1 and 4.2 show the idle probability.

Table 4.1: The Transient State Probabilities

N=20, L=9, M=10, λ=5, μ_i(i=1, 2, ... 10)=1, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9					
$p_{(i,j;t)}$	t=0.1	t=0.2	t=0.3	t=0.4	t=0.5
$p_{(0,0;t)}$	8.09636×10^{-05}	3.18357×10^{-08}	1.30281×10^{-10}	6.17838×10^{-12}	1.24395×10^{-12}
$p_{(0,1;t)}$	1.01094×10^{-03}	1.02241×10^{-06}	7.59276×10^{-09}	4.86133×10^{-10}	1.09355×10^{-10}
$p_{(0,2;t)}$	5.93189×10^{-03}	1.50809×10^{-05}	1.96732×10^{-07}	1.66146×10^{-08}	4.14725×10^{-09}
$p_{(0,3;t)}$	2.17513×10^{-02}	1.36035×10^{-04}	3.02637×10^{-06}	3.30082×10^{-07}	9.09015×10^{-08}
$p_{(0,4;t)}$	5.59062×10^{-02}	8.42640×10^{-04}	3.10994×10^{-05}	4.29673×10^{-06}	1.29935×10^{-06}
$p_{(0,5;t)}$	1.07076×10^{-01}	3.81395×10^{-03}	2.27409×10^{-04}	3.91025×10^{-05}	1.29382×10^{-05}
$p_{(0,6;t)}$	1.58584×10^{-01}	1.30976×10^{-02}	1.22879×10^{-03}	2.58809×10^{-04}	9.34783×10^{-05}
$p_{(0,7;t)}$	1.85996×10^{-01}	3.49437×10^{-02}	5.01998×10^{-03}	1.27747×10^{-03}	5.03223×10^{-04}
$p_{(0,8;t)}$	1.75455×10^{-01}	7.34125×10^{-02}	1.56978×10^{-02}	4.77536×10^{-03}	2.05358×10^{-03}
$p_{(0,9;t)}$	1.34318×10^{-01}	1.21766×10^{-01}	3.76847×10^{-02}	1.36227×10^{-02}	6.41819×10^{-03}
$p_{(0,10;t)}$	8.30486×10^{-02}	1.56954×10^{-01}	6.88885×10^{-02}	2.96581×10^{-02}	1.54246×10^{-02}
$p_{(1,10;t)}$	4.30139×10^{-02}	1.71013×10^{-01}	1.08150×10^{-01}	5.65055×10^{-02}	3.29169×10^{-02}
$p_{(2,10;t)}$	1.86098×10^{-02}	1.57556×10^{-01}	1.45992×10^{-01}	9.41443×10^{-02}	6.21872×10^{-02}
$p_{(3,10;t)}$	6.68068×10^{-03}	1.22039×10^{-01}	1.68443×10^{-01}	1.36146×10^{-01}	1.03081×10^{-01}
$p_{(4,10;t)}$	1.96808×10^{-03}	7.85983×10^{-02}	1.64180×10^{-01}	1.68714×10^{-01}	1.47866×10^{-01}
$p_{(5,10;t)}$	4.67930×10^{-04}	4.13757×10^{-02}	1.32792×10^{-01}	1.75835×10^{-01}	1.80055×10^{-01}
$p_{(6,10;t)}$	8.75968×10^{-05}	1.73560×10^{-02}	8.68268×10^{-02}	1.50073×10^{-01}	1.81191×10^{-01}
$p_{(7,10;t)}$	1.24321×10^{-05}	5.58212×10^{-03}	4.41475×10^{-02}	1.00920×10^{-01}	1.44999×10^{-01}
$p_{(8,10;t)}$	1.25767×10^{-06}	1.29510×10^{-03}	1.64603×10^{-02}	5.05137×10^{-02}	8.71322×10^{-02}
$p_{(9,10;t)}$	8.29812×10^{-08}	2.02527×10^{-04}	4.22747×10^{-03}	1.75120×10^{-02}	3.60646×10^{-02}
Total Probability	0.91695	1.00000	1.00000	0.999999	1.00000

Table 4.2: The Transient State Probabilities

N=20, L=9, M=10, λ=5, μ_i(i=1, 2, ... 10)=1, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9					
$p_{(i,j;t)}$	t=0.6	t=0.7	t=0.8	t=0.9	t=1
$p_{(0,0;t}$	4.69022×10^{-13}	2.45094×10^{-13}	1.58354×10^{-13}	1.18682×10^{-13}	9.86076×10^{-14}
$p_{(0,1;t)}$	4.32189×10^{-11}	2.32105×10^{-11}	1.52749×10^{-11}	1.15948×10^{-11}	9.71692×10^{-12}
$p_{(0,2;t)}$	1.71702×10^{-09}	9.48096×10^{-10}	6.35831×10^{-10}	4.88973×10^{-10}	4.13393×10^{-10}
$p_{(0,3;t)}$	3.94149×10^{-08}	2.23911×10^{-08}	1.53108×10^{-08}	1.19330×10^{-08}	1.01796×10^{-08}
$p_{(0,4;t)}$	5.90154×10^{-07}	3.45201×10^{-07}	2.40828×10^{-07}	1.90308×10^{-07}	1.63851×10^{-07}
$p_{(0,5;t)}$	6.15940×10^{-06}	3.71361×10^{-06}	2.64541×10^{-06}	2.12059×10^{-06}	1.84325×10^{-06}
$p_{(0,6;t)}$	4.67000×10^{-05}	2.90615×10^{-05}	2.11592×10^{-05}	1.72163×10^{-05}	1.51132×10^{-05}
$p_{(0,7;t)}$	2.64322×10^{-04}	1.70074×10^{-04}	1.26714×10^{-04}	1.04727×10^{-04}	9.28860×10^{-05}
$p_{(0,8;t)}$	1.13732×10^{-03}	7.58365×10^{-04}	5.79059×10^{-04}	4.86578×10^{-04}	4.36259×10^{-04}
$p_{(0,9;t)}$	3.76345×10^{-03}	2.60848×10^{-03}	2.04516×10^{-03}	1.74923×10^{-03}	1.58645×10^{-03}
$p_{(0,10;t)}$	9.63558×10^{-03}	6.97065×10^{-03}	5.62609×10^{-03}	4.90528×10^{-03}	4.50402×10^{-03}
$p_{(1,10;t)}$	2.21170×10^{-02}	1.67982×10^{-02}	1.40055×10^{-02}	1.24726×10^{-02}	1.16075×10^{-02}
$p_{(2,10;t)}$	4.52822×10^{-02}	3.62633×10^{-02}	3.13041×10^{-02}	2.85085×10^{-02}	2.69065×10^{-02}
$p_{(3,10;t)}$	8.18687×10^{-02}	6.93704×10^{-02}	6.21099×10^{-02}	5.78887×10^{-02}	5.54274×10^{-02}
$p_{(4,10;t)}$	1.28837×10^{-01}	1.15860×10^{-01}	1.07743×10^{-01}	1.02833×10^{-01}	9.99070×10^{-02}
$p_{(5,10;t)}$	1.73052×10^{-01}	1.65614×10^{-01}	1.60150×10^{-01}	1.56587×10^{-01}	1.54380×10^{-01}
$p_{(6,10;t)}$	1.93100×10^{-01}	1.97150×10^{-01}	1.98422×10^{-01}	1.98785×10^{-01}	1.98869×10^{-01}
$p_{(7,10;t)}$	1.72194×10^{-01}	1.87908×10^{-01}	1.96906×10^{-01}	2.02086×10^{-01}	2.05087×10^{-01}
$p_{(8,10;t)}$	1.15661×10^{-01}	1.34905×10^{-01}	1.47036×10^{-01}	1.54428×10^{-01}	1.58853×10^{-01}
$p_{(9,10;t)}$	5.30338×10^{-02}	6.55897×10^{-02}	7.39230×10^{-02}	7.91476×10^{-02}	8.23251×10^{-02}
Total Probability	0.99999	0.99999	1.00000	1.00000	0.99999

Table 4.3: The system performance measures

$N=20, L=9, M=10, \lambda=5, \mu_i(i=1,2,...10)=1,1.1,1.2,1.3,1.4,1.5,1.6,1.7,1.8,1.9$					
t	0.1	0.2	0.3	0.4	0.5
$L_1(t)$	0.11119	1.52891	3.22587	4.37586	5.08205
$L_2(t)$	6.64479	5.12772	4.42328	4.84396	5.31464
$L_3(t)$	7.24202	9.54899	9.90960	9.97174	9.98752
$L_4(t)$	12.49300	8.17015	5.92443	4.67239	3.93952
$E(W(t))$	0.02224	0.30578	0.64517	0.87517	1.01641
t	0.6	0.7	0.8	0.9	1.0
$L_1(t)$	5.50545	5.75675	5.90513	5.99249	6.04380
$L_2(t)$	5.64694	5.85739	5.98546	6.06203	6.10736
$L_3(t)$	9.99295	9.99523	9.99633	9.99690	9.99719
$L_4(t)$	3.50683	3.25158	3.10133	3.01301	2.96114
$E(W(t))$	1.10109	1.15135	1.18103	1.19850	1.20876

In the figure 4.4 for varying values of t, the mean length $L_1(t), L_2(t), L_3(t), L_4(t)$ are drawn as graphs. In the figure 4.5, the graph of expected waiting time using Little's Law are drawn.

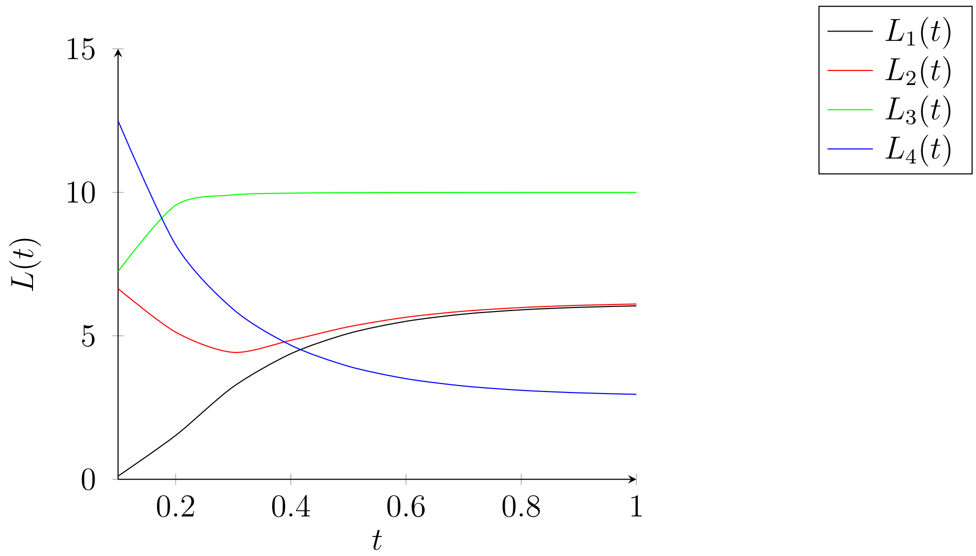

Figure: 4.4 Mean number of customers

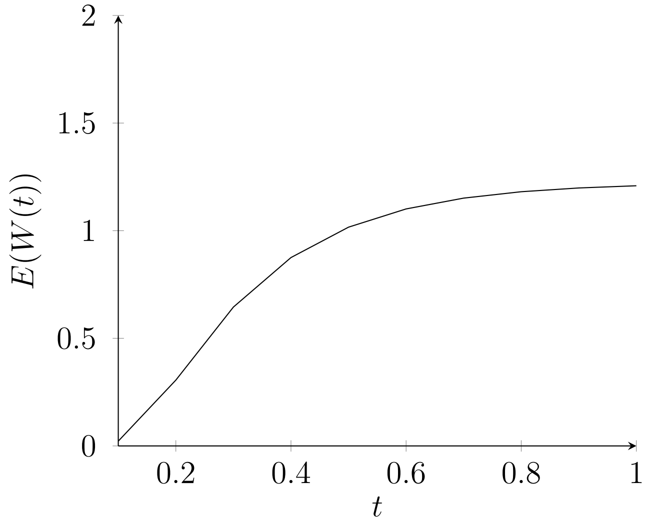

Figure: 4.5 Mean waiting time

4.2 Steady State Case

we calculated the stationary probabilities and the performance measures obtained in subsection 2.5.2. For the analysis, we vary the arrival rate λ from 1 to 10. The steady state probabilites are presented in tables 4.6 and 4.7. The corresponding system performance measures are presented in tables 4.8. In the figure 4.9 the system performance measures L_1, L_2, L_3 , L_4 and L_5 are shown as graphs and the mean waiting time is shown in graph 4.10

Table 4.6: The Steady State Probabilities

N=20, L=9, M=10, μ_i(i=1, 2, ... 10)=1, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9					
p_i	λ=1	λ=2	λ=3	λ=4	λ=5
$p_{(0,0)}$	1.47986×10^{-05}	2.48021×10^{-08}	1.57922×10^{-10}	2.43566×10^{-12}	7.35657×10^{-14}
$p_{(0,1)}$	2.95972×10^{-04}	9.92082×10^{-07}	9.47534×10^{-09}	1.94853×10^{-10}	7.35657×10^{-12}
$p_{(0,2)}$	2.55612×10^{-03}	1.71360×10^{-05}	2.45497×10^{-07}	6.73128×10^{-09}	3.17670×10^{-10}
$p_{(0,3)}$	1.27806×10^{-02}	1.71360×10^{-04}	3.68246×10^{-06}	1.34626×10^{-07}	7.94175×10^{-09}
$p_{(0,4)}$	4.17827×10^{-02}	1.12043×10^{-03}	3.61164×10^{-05}	1.76049×10^{-06}	1.29817×10^{-07}
$p_{(0,5)}$	9.55033×10^{-02}	5.12196×10^{-03}	2.47656×10^{-04}	1.60959×10^{-05}	1.48362×10^{-06}
$p_{(0,6)}$	1.59172×10^{-01}	1.70732×10^{-02}	1.23828×10^{-03}	1.07306×10^{-04}	1.23635×10^{-05}
$p_{(0,7)}$	1.98965×10^{-01}	4.26830×10^{-02}	4.64354×10^{-03}	5.36530×10^{-04}	7.72721×10^{-05}
$p_{(0,8)}$	1.90187×10^{-01}	8.15998×10^{-02}	1.33160×10^{-02}	2.05144×10^{-03}	3.69315×10^{-04}
$p_{(0,9)}$	1.40880×10^{-01}	1.20889×10^{-01}	2.95912×10^{-02}	6.07833×10^{-03}	1.36783×10^{-03}
$p_{(0,10)}$	8.15618×10^{-02}	1.39976×10^{-01}	5.13953×10^{-02}	1.40761×10^{-02}	3.95952×10^{-03}
$p_{(1,10)}$	4.29273×10^{-02}	1.47343×10^{-01}	8.11504×10^{-02}	2.96340×10^{-02}	1.04198×10^{-02}
$p_{(2,10)}$	2.03340×10^{-02}	1.39589×10^{-01}	1.15319×10^{-01}	5.61486×10^{-02}	2.46785×10^{-02}
$p_{(3,10)}$	8.56168×10^{-03}	1.17548×10^{-01}	1.45666×10^{-01}	9.45660×10^{-02}	5.19546×10^{-02}
$p_{(4,10)}$	3.15430×10^{-03}	8.66145×10^{-02}	1.60999×10^{-01}	1.39360×10^{-01}	9.57059×10^{-02}
$p_{(5,10)}$	9.96095×10^{-04}	5.47039×10^{-02}	1.52526×10^{-01}	1.76034×10^{-01}	1.51115×10^{-01}
$p_{(6,10)}$	2.62130×10^{-04}	2.87915×10^{-02}	1.20415×10^{-01}	1.85299×10^{-01}	1.98835×10^{-01}
$p_{(7,10)}$	5.51853×10^{-05}	1.21227×10^{-02}	7.60516×10^{-02}	1.56041×10^{-01}	2.09300×10^{-01}
$p_{(8,10)}$	8.71347×10^{-06}	3.82824×10^{-03}	3.60245×10^{-02}	9.85525×10^{-02}	1.65237×10^{-01}
$p_{(9,10)}$	9.17208×10^{-07}	8.05945×10^{-04}	1.13761×10^{-02}	4.14958×10^{-02}	8.69668×10^{-02}
Total Probability	0.999985	1	1	1	1

Table 4.7: The Steady State Probabilities

N=20, L=9, M=10, μ_i(i=1, 2, ... 10)=1, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9					
p_i	λ=6	λ=7	λ=8	λ=9	λ=10
$p_{(0,0)}$	3.68915×10^{-15}	2.72596×10^{-16}	2.72682×10^{-17}	3.47436×10^{-18}	5.39147×10^{-19}
$p_{(0,1)}$	4.42698×10^{-13}	3.81634×10^{-14}	4.36292×10^{-15}	6.25384×10^{-16}	1.07829×10^{-16}
$p_{(0,2)}$	2.29398×10^{-11}	2.30715×10^{-12}	3.01438×10^{-13}	4.86094×10^{-14}	9.31254×10^{-15}
$p_{(0,3)}$	6.88194×10^{-10}	8.07503×10^{-11}	1.20575×10^{-11}	2.18742×10^{-12}	4.65627×10^{-13}
$p_{(0,4)}$	1.34992×10^{-08}	1.84794×10^{-09}	3.15351×10^{-10}	6.43607×10^{-11}	1.52224×10^{-11}
$p_{(0,5)}$	1.85132×10^{-07}	2.95670×10^{-08}	5.76641×10^{-09}	1.32399×10^{-09}	3.47941×10^{-10}
$p_{(0,6)}$	1.85132×10^{-06}	3.44949×10^{-07}	7.68855×10^{-08}	1.98599×10^{-08}	5.79902×10^{-09}
$p_{(0,7)}$	1.38849×10^{-05}	3.01830×10^{-06}	7.68855×10^{-07}	2.23424×10^{-07}	7.24877×10^{-08}
$p_{(0,8)}$	7.96339×10^{-05}	2.01960×10^{-05}	5.87948×10^{-06}	1.92210×10^{-06}	6.92897×10^{-07}
$p_{(0,9)}$	3.53929×10^{-04}	1.04720×10^{-04}	3.48414×10^{-05}	1.28140×10^{-05}	5.13257×10^{-06}
$p_{(0,10)}$	1.22944×10^{-03}	4.24391×10^{-04}	1.61370×10^{-04}	6.67677×10^{-05}	2.97149×10^{-05}
$p_{(1,10)}$	3.88243×10^{-03}	1.56355×10^{-03}	6.79455×10^{-04}	3.16268×10^{-04}	1.56394×10^{-04}
$p_{(2,10)}$	1.10343×10^{-02}	5.18439×10^{-03}	2.57477×10^{-03}	1.34830×10^{-03}	7.40815×10^{-04}
$p_{(3,10)}$	2.78761×10^{-02}	1.52803×10^{-02}	8.67293×10^{-03}	5.10935×10^{-03}	3.11922×10^{-03}
$p_{(4,10)}$	6.16208×10^{-02}	3.94071×10^{-02}	2.55623×10^{-02}	1.69415×10^{-02}	1.14919×10^{-02}
$p_{(5,10)}$	1.16755×10^{-01}	8.71105×10^{-02}	6.45785×10^{-02}	4.81496×10^{-02}	3.62901×10^{-02}
$p_{(6,10)}$	1.84350×10^{-01}	1.60467×10^{-01}	1.35955×10^{-01}	1.14039×10^{-01}	9.55002×10^{-02}
$p_{(7,10)}$	2.32864×10^{-01}	2.36477×10^{-01}	2.28976×10^{-01}	2.16073×10^{-01}	2.01053×10^{-01}
$p_{(8,10)}$	2.20608×10^{-01}	2.61370×10^{-01}	2.89233×10^{-01}	3.07051×10^{-01}	3.17452×10^{-01}
$p_{(9,10)}$	1.39331×10^{-01}	1.92588×10^{-01}	2.43565×10^{-01}	2.90891×10^{-01}	3.34160×10^{-01}
Total Probability	1	1	1	1	1

Table 4.8: The system performance measures

$N = 20, L = 9, M = 10, \mu_i(i = 1, 2, ...10) = 1, 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 1.9$					
λ	1	2	3	4	5
L_1	0.12892	1.69463	3.80085	5.22921	6.11674
L_2	6.77014	5.26705	4.72898	5.44558	6.17223
L_3	50.69000	38.3235	28.1986	33.7831	41.4538
L_4	7.93037	20.5187	38.9366	52.5085	61.2229
L_5	13.12460	9.48792	5.83877	3.93266	2.92708
$E(W)$	0.01509	0.12768	0.25009	0.32739	0.37447
λ	6	7	8	9	10
L_1	6.69482	7.09334	7.3815	7.59802	7.76585
L_2	6.71105	7.09871	7.38348	7.59882	7.7662
L_3	47.8128	52.7119	56.4858	59.4407	61.7993
L_4	66.9645	70.9388	73.81700	75.98100	77.6588
L_5	2.31848	1.91118	1.62020	1.40268	1.23446
$E(W)$	0.40497	0.42596	0.44113	0.45253	0.46136

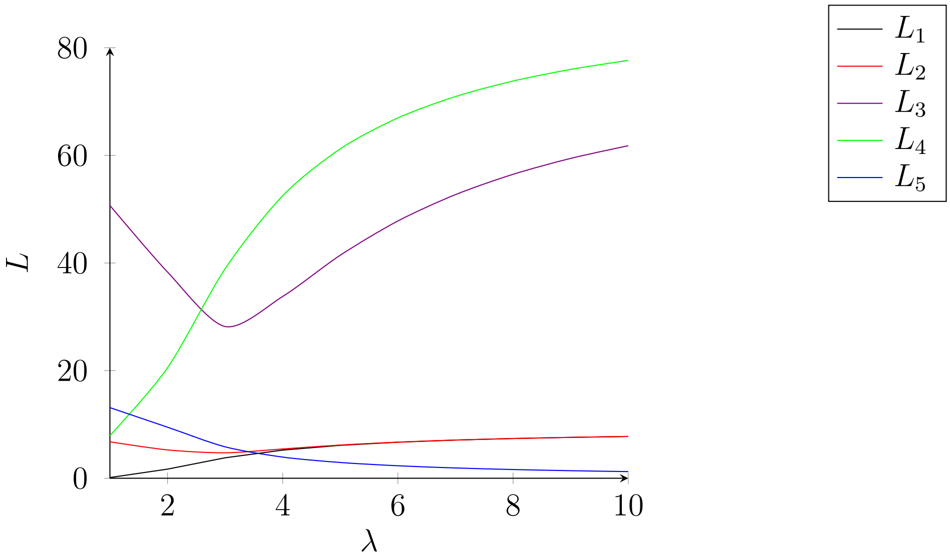

Figure: 4.9 Mean number of customers

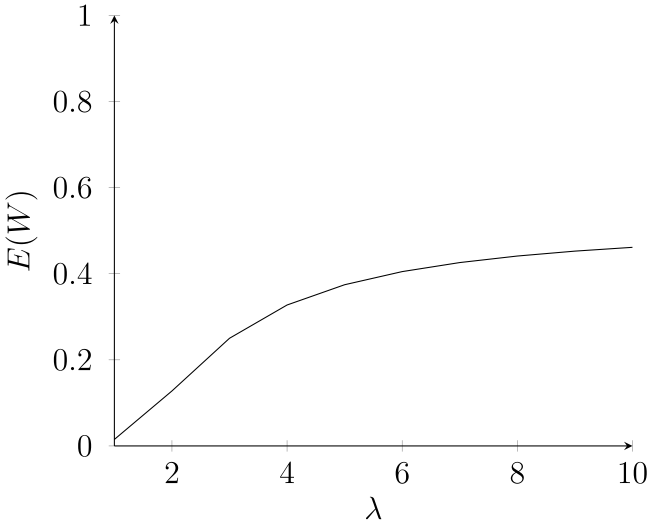

Figure: 4.10 Mean waiting time

5 Control Chart Analysis

In this section, the statistical process control is carried out using control chart analysis. Several types of control charts are available in the literature, but whatever may be the type, all have some few common characters. They contains the upper and lower control limits within which all observations should lie, if the process is in control. There is a central line(CL), which is usually considered to be the target value for the process, they generally show the numbers along the vertical axis to define the values of the control limits and if the observations are beyond these points, the charts may be tailored to suit the requirements of the process. A typical control chart has all points lies nearly within the upper control limit(UCL) and the lower control limit(LCL). The control chart for number of customers in the system are obtained using the following control limits:

- $CL = L_2$ (5.1)
- $UCL = L_2 + 3\sqrt{V}$ (5.2)
- $LCL = L_2 - 3\sqrt{V}$ (5.3)

where, $V = L_3 - L_2^2$

The UCL and LCL should be symmetric around CL. But in some cases, the LCL becomes negative. In this case, the LCL can be rounded to zero. The numerical results are obtained adhere with the stability condition. Our computational experience shows that the number of customers in all the samples are distributed between UCL and LCL, in particular very nearest to CL. In this study we never experience that the situation is out of control.

Table 5.1: The Control Chart Table(μ =1.9)

λ	LCL	CL	UCL
1	0.15978	6.77014	13.38050
2	-4.49180≈0	5.26705	15.02590
3	-2.51796≈0	4.72898	11.97592
4	-0.65022≈0	5.44558	11.54138
5	0.67529	6.17223	11.66917
6	1.71390	6.71105	11.70820
7	2.52903	7.09871	11.66839
8	3.17275	7.38348	11.59421
9	3.68887	7.59882	11.50877
10	4.10984	7.76620	11.42256

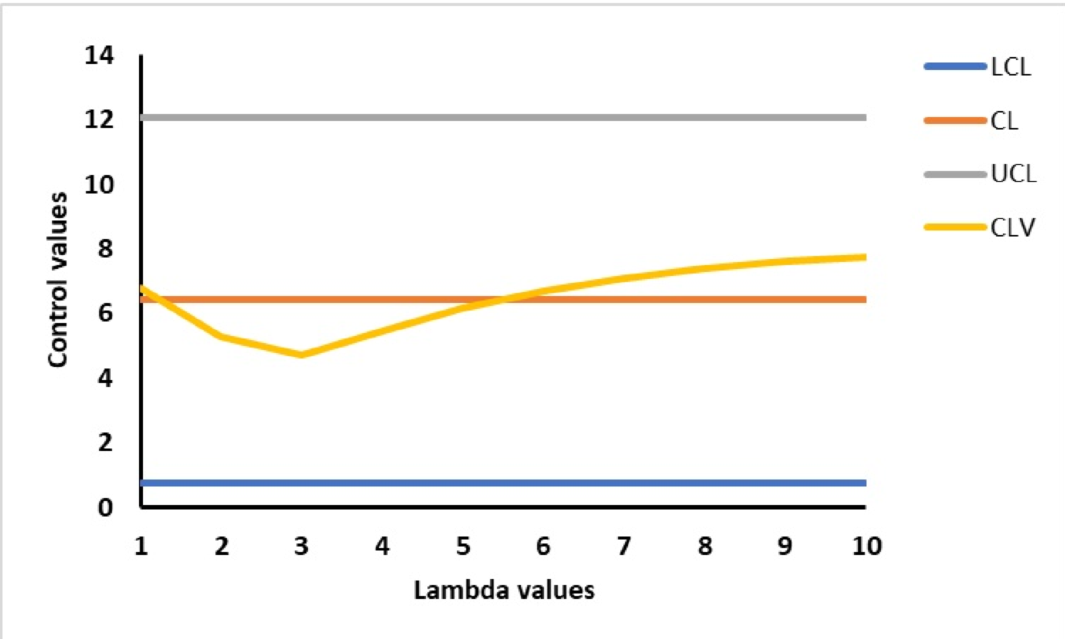

Figure 5.2: Control chart for number of customers(μ =1.9)

Table 5.3: The Control Chart Table(μ =2.0)

λ	LCL	CL	UCL
1	0.34063	6.8039	13.26717
2	-4.49873≈0	5.43910	15.37693
3	-2.88420≈0	4.67298	12.23016
4	-0.96340≈0	5.28363	11.53066
5	0.37418	6.00790	11.64162
6	1.42823	6.56775	11.70727
7	2.26583	6.97691	11.68799
8	2.93291	7.27946	11.62601
9	3.47014	7.50892	11.54771
10	3.90959	7.68750	11.46541

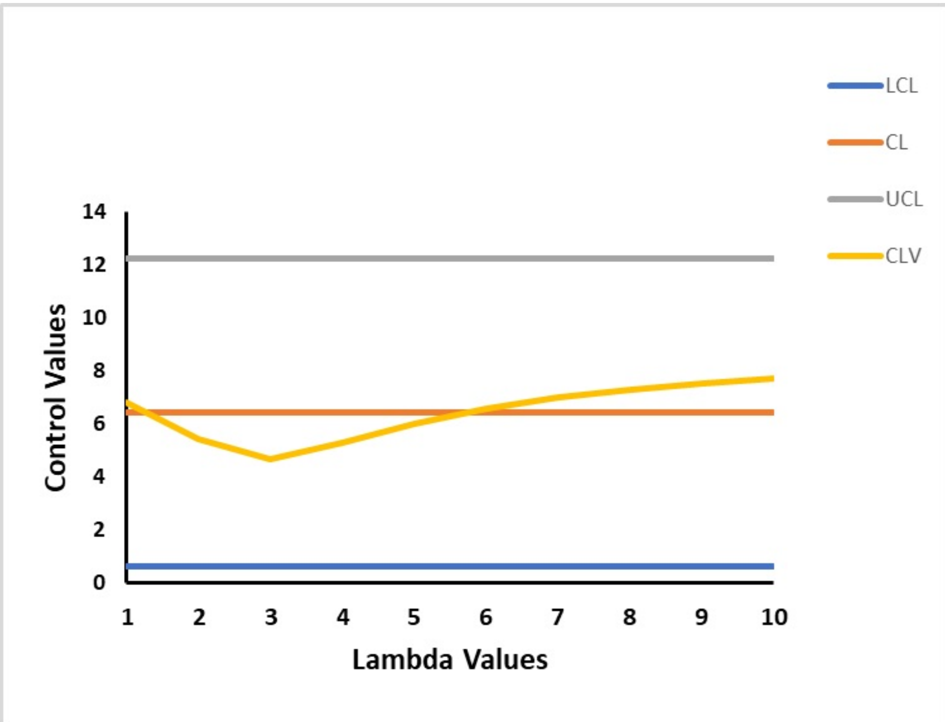

Figure 5.4: Control chart for number of customers(μ =2.0)

A numerical model is provided to analyze the performance of the queue using control chart analysis with reference to difference arrival rates, production times(service times), and waiting space size. The corresponding graphs are drawn and are given in Figures 5.2 and 5.4. The values of CL, UCL and LCL are obtained from the formulas in the equations (5.1),(5.2) and (5.3). The values are also tabulated and are shown in the tables 5.1 and 5.3. The UCL and LCL should be symmetric around CL. But in some cases, the LCL becomes negative. In this case, the LCL can be rounded to zero. Our computational experience shows that the number of customers in all the samples is distributed between UCL and LCL, in particular very nearest to CL. In this study we never experience that the situation is out of control.

6 Conclusion

In this paper, control limits have been obtained for number of customers in the system of the given queueing model. Numerical illustrations are provided. Our computational experience shows that, the process control shows to be under control.

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