A Mathematical Models Using COM-Poisson Thomas Process

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ABSTRACT

COM-Poisson Process is the generalization of Poisson Process. COM-Poisson Thomas process is a generalization of Thomas distribution. COM-Poisson Thomas Process is a compound COM-Poisson process with compounding shifted Poisson distribution. In this paper, some mathematical models like Traffic accidents and fatalities model, shock model, natural disaster model, game model and system reliability models using COM-Poisson Thomas Process are framed.

Keywords: COM-Poisson Thomas Process, Inter-arrival Time, Conditional distribution, Uniform distribution, Shock model, Natural disaster model, Game model, software reliability model.

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1 Introduction

The COM-Poisson distribution was proposed by Conway and Maxwell(1962). He used it to model queuing systems with state-dependent service times. Shmueli et al (2005) revived the COM-Poisson distribution and showed that this distribution is best suited for over and under dispersed data. Initially, this distribution is used to model the number of purchases by customers at an online grocery store, where the data showed different levels of dispersion for different product categories. The original development started from the point of allowing the ratio of consecutive probabilities $P(U = u\ 1)/P(U = u)$ to be more flexible than a linear function of u, as dictated by a Poisson distribution.

The COM-Poisson distribution is a two-parameter generalization of the Poisson distribution. It also generalizes the Bernoulli and geometric distributions and it belongs to the exponential family and to the two parameter power series distribution family.

In 2018, Priyadharshini et al derived the COM-Poisson Process and the mathematical models using COM-Poisson Process are derived.

In 2019, Priyadharshini et al introduced a COM-Poisson Thomas distribution, which is a compound COM-Poisson distribution with compounding shifted Poisson distribution.

In this paper, some mathematical models like Traffic accidents and fatalities model, shock model, natural disaster model, game model and system reliability models using COM-Poisson Thomas Process are framed.

This paper is organised as follows: In section 2, COM-Poisson Thomas Process is studied and some of the properties of COM-Poisson Thomas process are derived. All the mathematical models are derived in section 3. Conclusion is given in Section 4.

2 COM-Poisson Thomas Process

The COM-Poisson Thomas process is compound COM-Poisson process with compounding shifted Poisson distribution.
Then, the COM-Poisson Thomas process is derived by assuming the following.

(i) X follows shifted Poisson distribution with parameter ψ. (ie).,

$$X \sim Shifted\ Poisson(\psi)$$

(ii) $Y(t)$ follows COM-Poisson process with parameters μ and γ. (ie).,

$$Y(t) \sim COM - Poisson(\mu, \gamma)$$
$$N(t) = X_1 + X_2 + \ldots + X_{Y(t)}$$

The random variable $N(t)$ formed by compounding these two random variables X and $Y(t)$ gives the COM-Poisson Thomas process with parameters $\mu > 0$, $\gamma \geq 0$ and $\psi > 0$.
It is denoted by $N(t) \sim CPTP(\mu, \gamma, \psi)$.

2.1 Definition

A counting process $\{N(t), t \geq 0\}$ is said to be a COM-Poisson Thomas process with parameters μ, γ and ψ if

(i) it starts at zero, N(0) = 0.

(ii) $N(t)$ is a process with independent increments.

(iii) for each $t > 0$, the number of arrivals $N(t)$ in any interval of length t is COM-Poisson Thomas distributed with parameters μ, γ and ψ

$$P(N(t) = n) = \begin{cases} \dfrac{1}{Z(\mu t, \gamma)} & \text{for } n = 0 \\ \dfrac{1}{Z(\mu t, \gamma)} \displaystyle\sum_{j=1}^{n} \frac{(\mu t e^{-\psi})^j (j\psi)^{n-j}}{(j!)^{\gamma} (n-j)!} & \text{for } n = 1, 2, \ldots \end{cases} \quad (2.1)$$

2.2 Properties

2.2.1 Inter-arrival Time

Let $\{N(t), t \geq 0\}$ follows COM-Poisson Thomas Process, and let X be the random variable representing the interval between two successive occurrences of $\{N(t), t \geq 0\}$ and let $Pr\{X \leq x\} = G(x)$ be its distribution function.
Let us denote two successive events by A_i and A_{i+1} and assume that A_i occured at the instant t_i. Then

$$\begin{aligned} Pr\{X > x\} &= Pr\{A_{i+1} \text{did not occur in } (t_i, t_{i+1}) \text{ given that } A_i \text{ occured at the instant } t_i\} \\ &= Pr\{A_{i+1} \text{did not occur in } (t_i, t_{i+1})/N(t_i) = i\} \\ &= Pr\{\text{no occurrences takes place in an interval } (t_i, t_{i+1}) \text{ of length } x/N(t_i) = i\} \\ &= Pr\{N(x) = 0/N(t_i) = i\} = p_0(x) = \frac{1}{Z(\mu x, \gamma)}, x > 0 \end{aligned}$$

Since i is arbitrary, the interval X between any two successive occurrences becomes,

$$G(x) = Pr\{X \leq x\} = 1 - Pr\{X > x\} = 1 - \frac{1}{Z(\mu x, \gamma)}, x > 0.$$

The density function is

$$g(u) = G'(u) = \frac{Z_x(\mu x, \gamma)}{[Z(\mu x, \gamma)]^2}, x > 0$$

2.2.2 Conditional distribution of the arrival times

Assume that exactly one event of a COM-Poisson Thomas process has taken place by time t, and let us determine the distribution of the time at which the event occured.
For $s < t$,

$$P\{X_1 < s/N(t) = 1\} = \frac{P\{X_1 < s, N(t) = 1\}}{P\{N(t) = 1\}}$$

$$= \frac{P\{\text{1 event in [0,s), 0 events in [s,t)}\}}{P\{N(t) = 1\}}$$

$$= \frac{P\{\text{1 event in [0,s)}\}P\{\text{0 events in [s,t)}\}}{P\{N(t) = 1\}}$$

$$= \frac{\frac{\mu s e^{-\psi}}{Z(\mu s,\gamma)}\frac{1}{Z(\mu(t-s),\gamma)}}{\frac{\mu t e^{-\psi}}{Z(\mu t,\gamma)}}$$

$$= \frac{s}{t}\frac{Z(\mu t,\gamma)}{Z(\mu s,\gamma)Z(\mu(t-s),\gamma)}$$

Proof of generalization

Let $U_1, U_2, ..., U_n$ be n random variables. $U_{(1)}, U_{(2)}, ..., U_{(n)}$ are the order statistics corresponding to $U_1, U_2, ..., U_n$ if $U_{(k)}$ is the k^{th} smallest value among $U_1, U_2, ..., U_n$, $k = 1, 2, ..., n$. If the $U_i^{\prime}s$ are independent identically distributed continuous random variables with probability density g, then the joint density of the order statistics $U_{(1)}, U_{(2)}, ..., U_{(n)}$ is given by

$$g(U_1, U_2, ..., U_n) = n!\prod_{i=1}^{n} g(U_i), \quad U_1 < U_2 < ... < U_n$$

The above follows since

(i) $U_{(1)}, U_{(2)}, ..., U_{(n)}$ will equal $(u_1, u_2, ..., u_n)$ if $(U_1, U_2, ..., U_n)$ is equal to any of the $n!$ permutations of $(u_1, u_2, ..., u_n)$ and

(ii) the probability density that $(U_1, U_2, ..., U_n)$ is equal to $u_{i_1}, u_{i_2}, ..., u_{i_n}$ is $g(u_{i_1})g(u_{i_2})...g(u_{i_n}) = \prod_{i=1}^{n} g(u_i)$ when $(u_{i_1}, u_{i_2}, ..., u_{i_n})$ is a permutation of $(u_1, u_2, ..., u_n)$

Let $V_1, V_2, ..., V_n$ be n arrival times have the same distribution as the order statistics corresponding to n independent random variables.
To determine the conditional density function of $V_1, V_2, ..., V_n$ given that $N(t) = n$. Let $0 < t_1 < t_2 < ... < t_{n+1} = t$ and let h_i be small enough so that $t_i + h_i < t_{i+1}, i = 1, 2, ...n$.
Now,

$$P\{t_i \le V_i \le t_i + h_i, i = 1, 2, ..., n/N(t) = n\} =$$

$$\frac{P\{\text{ exactly 1 event in } [t_i, t_i + h_i], i = 1, 2, ...n, \text{ no events elsewhere in } [0, t]\}}{P\{N(t) = n\}}$$

$$= \frac{\frac{\mu h_1 e^{-\psi}}{Z(\mu h_1,\gamma)}\frac{\mu h_2 e^{-\psi}}{Z(\mu h_2,\gamma)}\cdots\frac{\mu h_n e^{-\psi}}{Z(\mu h_n,\gamma)}\frac{1}{Z(\mu(t-h_1-h_2-...-h_n),\gamma)}}{\frac{1}{Z(\mu t,\gamma)}\sum_{j=1}^{n}\frac{(\mu t e^{-\psi})^j (j\psi)^{n-j}}{(j!)^\gamma (n-j)!}}$$

$$= \frac{\mu^n e^{-n\psi}}{\frac{1}{Z(\mu t,\gamma)}\sum_{j=1}^{n}\frac{(\mu t e^{-\psi})^j (j\psi)^{n-j}}{(j!)^\gamma (n-j)!}} \times \frac{Z(\mu t,\gamma)}{Z(\mu h_1,\gamma)Z(\mu h_2,\gamma)...Z(\mu h_n,\gamma)Z(\mu(t-h_1-h_2-...-h_n),\gamma)} h_1 h_2 ... h_n$$

Hence

$$\frac{P\{t_i \le V_i \le t_i + h_i,\ i = 1, \ldots, n/N(t) = n\}}{h_1 h_2 \ldots h_n} = \frac{\mu^n e^{-n\psi}}{\frac{1}{Z(\mu t,\gamma)} \sum_{j=1}^{n} \frac{(\mu t e^{-\psi})^j (j\psi)^{n-j}}{(j!)^\gamma (n-j)!}} \times \frac{Z(\mu t,\gamma)}{Z(\mu h_1,\gamma)\ldots Z(\mu h_n,\gamma) Z(\mu(t - h_1 - h_2 - \ldots - h_n),\gamma)}$$

and by taking the $h_i \to 0$, we obtain that the conditional density of $V_1, V_2, \ldots, V_n$ given that $N(t) = n$ is

$$g(t_1, t_2, \ldots, t_n) = \frac{\mu^n e^{-n\psi}}{\frac{1}{Z(\mu t,\gamma)} \sum_{j=1}^{n} \frac{(\mu t e^{-\psi})^j (j\psi)^{n-j}}{(j!)^\gamma (n-j)!}}, \quad 0 < t_1 < t_2 < \ldots < t_n$$

3 Mathematical Models

3.1 Shock Model

Assumptions

(i) Suppose that a device is subjected to shocks that occur in accordance with a COM-Poisson Thomas process having rate $\frac{\mu(1+\psi)Z_\mu(\mu t,\gamma)}{Z(\mu t,\gamma)}$.

(ii) The i^{th} shock gives a damage D_i. The D_i, $i = 1$, are assumed to be independent and identically distributed and also to be independent of $\{N(t), t \ge 0\}$ where $N(t)$ denotes the number of shocks in $[0, t]$.

(iii) The damage due to a shock is assumed to decrease exponentially in time. That is, if a shock has an initial damage D, then a time t later its damage is $De^{-\alpha t}$.

(iv) Assume that the damages are additive, the damage at time t, $D(t)$ can be expressed as

$$D(t) = \sum_{i=1}^{N(t)} D_i e^{-\alpha(t-v_i)}$$

where v_i represent the arrival time of the i^{th} shock.

To determine $E[D(t)]$ as follows:

$$\begin{aligned}
E[D(t)/N(t) = n] &= E\left[\sum_{i=1}^{N(t)} D_i e^{-\alpha(t-v_i)} / N(t) = n\right] \\
&= E\left[\sum_{i=1}^{n} D_i e^{-\alpha(t-v_i)} / N(t) = n\right] \\
&= \sum_{i=1}^{n} E\left[D_i e^{-\alpha(t-v_i)} / N(t) = n\right] \\
&= \sum_{i=1}^{n} E\left[D_i/N(t) = n\right] E\left[e^{-\alpha(t-v_i)}/N(t) = n\right] \\
&= E[D] \sum_{i=1}^{n} E\left[e^{-\alpha(t-v_i)}/N(t) = n\right] \\
&= E[D] e^{-\alpha t} E\left[\sum_{i=1}^{n} e^{\alpha v_i}/N(t) = n\right]
\end{aligned}$$

Taking $U_1, \ldots, U_n$ be independent and identically distributed random variables, then

$$E\left[\sum_{i=1}^{n} e^{av_i}/N(t)=n\right] = E\left[\sum_{i=1}^{n} e^{aU_{(i)}}\right]$$

$$= E\left[\sum_{i=1}^{n} e^{aU_i}\right] = \frac{n}{t}\int_0^t e^{ax}dx = \frac{n}{at}(e^{at}-1)$$

Hence, $$E\left[D(t)/N(t)=n\right] = \frac{E[N(t)]}{at}(1-e^{-at})E[D]$$

$$E\left[D(t)\right] = \frac{\mu(1+\psi)Z_\mu(\mu t,\gamma)E[D]}{atZ(\mu t,\gamma)}(1-e^{-at})$$

3.2 Natural Disaster Model

Assumptions

(i) Assume that the occurrences of natural disasters follow a COM-Poisson Thomas process with parameters μ, γ and ψ.

(ii) Suppose that the time it takes to recover and rebuild after the n^{th} disaster is X_n, assume that $X_1, X_2, \ldots$ are independent random variables having common distribution functions $G(x) = P(X \le x)$.

(iii) Let Y_k is time to receive the k^{th} disaster.

(iv) There are N_T disasters up to time T.

The probability that everything is back to normal at time T is,

$$P\left(\max_{1\le i\le N_T}\{Y_i+X_i\}<T\right) = \sum_{n=0}^{\infty} P\left(\max_{1\le i\le N_T}\{Y_i+X_i\}<T \mid N_T=n\right) P(N_T=n)$$

$$= \sum_{n=0}^{\infty} P\left(\max_{1\le i\le N_T}\{Y_i+X_i\}<T \mid N_T=n\right) \times \frac{1}{Z(\mu t,\gamma)}\sum_{j=1}^{n}\frac{(\mu Te^{-\psi})^j(j\psi)^{n-j}}{(j!)^\gamma(n-j)!}$$

$$= \sum_{n=0}^{\infty} P\left(Y_1+X_1<T,\ldots,Y_n+X_n<T \mid N_T=n\right) \times \frac{1}{Z(\mu T,\gamma)}\sum_{j=1}^{n}\frac{(\mu Te^{-\psi})^j(j\psi)^{n-j}}{(j!)^\gamma(n-j)!}$$

$$= \sum_{n=0}^{\infty} n!P\left(U_1+X_1<T,\ldots,U_n+X_n<T\right) \frac{1}{Z(\mu T,\gamma)}\sum_{j=1}^{n}\frac{(\mu Te^{-\psi})^j(j\psi)^{n-j}}{(j!)^\gamma(n-j)!}$$

where $U_1, U_2, \ldots, U_n$ are independent and identically distributed on (0,T]

$$P\left(\max_{1\le i\le N_T}\{Y_i+X_i\}<T\right)=\sum_{n=0}^{\infty} n!\,[P\,(U_1+X_1<T)]^n\,\frac{1}{Z(\mu t,\gamma)}\sum_{j=1}^{n}\frac{(\mu Te^{-\psi})^j(j\psi)^{n-j}}{(j!)^\gamma(n-j)!}$$

$$=\sum_{n=0}^{\infty} n!\left(\int_0^T P\,(U_1+X_1<T\mid U_1=u)\,\frac{1}{T}du\right)^n\times\frac{1}{Z(\mu T,\gamma)}\sum_{j=1}^{n}\frac{(\mu Te^{-\psi})^j(j\psi)^{n-j}}{(j!)^\gamma(n-j)!}$$

$$=\sum_{n=0}^{\infty}\left(\int_0^T P\,(X_1<T-u\mid U_1=u)\,\frac{1}{T}du\right)^n\times\frac{1}{Z(\mu T,\gamma)}\sum_{j=1}^{n}\frac{(\mu Te^{-\psi})^j(j\psi)^{n-j}}{(j!)^\gamma(n-j)!}$$

$$=\sum_{n=0}^{\infty}\left(\int_0^T P\,(X_1<T-u)\,\frac{1}{T}du\right)^n\frac{1}{Z(\mu T,\gamma)}\sum_{j=1}^{n}\frac{(\mu Te^{-\psi})^j(j\psi)^{n-j}}{(j!)^\gamma(n-j)!}$$

$$=\sum_{n=0}^{\infty}\left(\frac{1}{T}\int_0^T G\,(T-u)\,du\right)^n\frac{1}{Z(\mu T,\gamma)}\sum_{j=1}^{n}\frac{(\mu Te^{-\psi})^j(j\psi)^{n-j}}{(j!)^\gamma(n-j)!}$$

$$=\sum_{n=0}^{\infty}\left(\frac{1}{\mu}\int_0^T G\,(z)\,dz\right)^n\frac{1}{Z(\mu T,\gamma)}\sum_{j=1}^{n}\frac{(\mu Te^{-\psi})^j(j\psi)^{n-j}}{(j!)^\gamma(n-j)!}$$

The average time to recover and rebuild from the natural disaster before T is $E\left(\max_{1\le i\le N_T}\{Y_i+X_i\}\right)$

Suppose $t\ge T$,

$\int$

$\int$

$P \quad \max \{Y_i \qquad \infty \quad P \quad \max \{Y_i + X_i\} < t \mid N_T = n \quad P(N_T = n)$

$= T \sum \quad 1 - \frac{1}{Z(\mu T, \gamma)} \; 1 \qquad \mu$

$+ X \} < t$

$= \qquad t-T$

$\leq i \leq N_T \qquad \leq i \leq N_T$

i

$n=0$

$\infty \quad \int t \quad n \quad n$

T $n=0$

Clearly, P max

$\leq i \leq N_T$

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$\{Y_i + X_i\} < t \quad = 0$

$$\sum_{n=0} .$$

for $t < T$

μ

$G(z)$ dz

$G(z)\,dz$ $_{j=1}$ $(j!)^{\gamma}(n-j)!$ dt

$t-T$

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$^{-\psi})$

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$\psi)^n$

$\sum$

$_{j=1}$ $(j!)^{\gamma}(n \quad j)!$ $-$

$-j$

E max

$1 \le i \le N_T$

∞

$$\{Y_i + X_i\} = P$$

0

max

$1 \leq i \leq N_T$

$\{Y_i + X_i\} > t$

dt

∞

$\max \{Y_i + X_i\} > t \quad dt$

$= T + \quad P$

$1 \le i \le N_T$

$\int^{T} \infty \quad \sum_{\infty} \quad \int t \quad n \quad \sum^{n} \overline{(\mu T e^{-\psi})^j (j\psi)^{n-j}}$

This can also be used as a model for insurance claims. Y_k is the time for the insurance company to receive the k^{th} claim and X_k is the time the insurance company takes to settle the claim.

3.3 Game Model

Assumptions

(i) Suppose team A and team B are engaging in a sport competition.

(ii) The points scored by team A follows a COM-Poisson Thomas process X_t with parameters λ, γ_1 and ψ_1 and the points scored by team B follows a COM-Poisson Thomas process Y_t with parameters μ, γ_2 and ψ_2.

(iii) Assume that the points scored by team A and the points scored by team B are independent. (ie)., X_t and Y_t are independent.

(iv) Let T be the duration of the competition.

The probability of game ties is

$$P(\text{Game ties}) = P(X_T = Y_T) = \sum_{k=1}^{\infty} P(X_T = k, Y_T = k)$$

$$= \sum_{k=0}^{\infty} \frac{1}{Z(\lambda t, \gamma_1)Z(\mu t, \gamma_2)} \sum_{j=1}^{k} \frac{(\lambda t e^{-\psi_1})^j (j\psi_1)^{k-j}}{(j!)^{\gamma_1}(k-j)!} \times \sum_{j=1}^{k} \frac{(\mu t e^{-\psi_2})^j (j\psi_2)^{k-j}}{(j!)^{\gamma_2}(k-j)!}$$

The probability of A wins is

$$P(\text{A wins}) = P(X_T > Y_T) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} P(X_T = k+l, Y_T = k)$$

$$= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{Z(\lambda t, \gamma_1)Z(\mu t, \gamma_2)} \sum_{j=1}^{k+l} \frac{(\lambda t e^{-\psi_1})^j (j\psi_1)^{k+l-j}}{(j!)^{\gamma_1}(k+l-j)!} \times \sum_{j=1}^{k} \frac{(\mu t e^{-\psi_2})^j (j\psi_2)^{k-j}}{(j!)^{\gamma_2}(k-j)!}$$

The probability of B wins is

$$P(\text{B wins}) = P(Y_T > X_T) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} P(Y_T = k+l, X_T = k)$$

$$= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{Z(\lambda t, \gamma_1)Z(\mu t, \gamma_2)} \sum_{j=1}^{k} \frac{(\lambda t e^{-\psi_1})^j (j\psi_1)^{k-j}}{(j!)^{\gamma_1}(k-j)!} \times \sum_{j=1}^{k+l} \frac{(\mu t e^{-\psi_2})^j (j\psi_2)^{k+l-j}}{(j!)^{\gamma_2}(k+l-j)!}$$

3.4 System reliability model using shock model approach

All the assumptions are same as in chapter 2.
The release time of the system is

$$W = X_0 + \sum_{i=1}^{N(t)} (X_i + Y_i)$$

where X_o is test phase time before the first fault direction.
The expected release time of the system is

$$E(W) = \mu_1 P(N(t) = 0) + \left(\mu_1 + \mu_2\right) + \frac{\mu_1}{a} + \frac{\mu_2}{b} + \frac{\mu_1}{a^2} + \frac{\mu_2}{b^2} + \ldots + \frac{\mu_1}{a^{k-1}} + \frac{\mu_2}{b^{k-1}} \quad P(N(t) = k)$$

Model 1

Let the test phase time of the system X_1 follows exponential distribution with parameter A, the fault removal time Y_1 follows exponential distribution with parameter B and the number of faults detections $N(t)$ follows COM-Poisson Thomas process with parameters μ, γ, ψ where

$$E(X_1) = \frac{1}{A}, \; A > 0$$

$$E(Y_1) = \frac{1}{B}, \; B > 0$$

$$E[N(t)] = \frac{\mu(1+\psi)Z_\mu(\mu t, \gamma)}{Z(\mu t, \gamma)}$$

Then

$$E(W) = \frac{1}{A}\frac{1}{Z(\mu t, \gamma)} + \left(\frac{1}{Z(\mu t, \gamma)}\sum_{j=1}^{k}\frac{\mu^j t^j e^{-j\psi}(j\psi)^{k-j}}{(j!)^\gamma (k-j)!}\right)\left(\sum_{i=1}^{k}\frac{1}{Aa^{i-1}} + \frac{1}{Bb^{i-1}}\right), \; k = 0, 1, 2, \ldots$$

Model 2

Let the testing phase of the system X_1 follows exponential distribution with parameter A, the fault removal time Y_1 follows Weibull distribution with parameter B, β and the number of faults detections $N(t)$ follows COM-Poisson Thomas process with parameters μ, γ, ψ. where

$$E(X_1) = \frac{1}{A}, \; A > 0$$

$$E(Y_1) = \frac{1}{B}, \; B > 0$$

$$E[N(t)] = \frac{\mu(1+\psi)Z_\mu(\mu t, \gamma)}{Z(\mu t, \gamma)}$$

Then

$$E(W) = \frac{1}{AZ(\mu t, \gamma)} + \left(\sum_{i=1}^{k}\frac{1}{Aa^{i-1}} + \frac{B\Gamma\left(\frac{1}{\beta}+1\right)}{b^{i-1}}\right)\left(\frac{1}{Z(\mu t, \gamma)}\sum_{j=1}^{k}\frac{\mu^j t^j e^{-j\psi}(j\psi)^{k-j}}{(j!)^\gamma (k-j)!}\right),$$

$$k = 0, 1, 2, \ldots$$

Model 3

Let the testing phase of the system X_1 follows Weibull distribution with parameter A, α, the fault removal time Y_1 follows exponential distribution with parameter B, β and the number of faults detections N follows COM-Poisson Thomas process with parameter μ, γ and ψ.
where

$$E(X_1) = A\Gamma\left(\frac{1}{\alpha} + 1\right), \; A > 0$$

$$E(Y_1) = \frac{1}{B}, \; B > 0$$

$$E[N(t)] = \frac{\mu(1+\psi)Z_\mu(\mu t, \gamma)}{Z(\mu t, \gamma)}$$

Then

$$E(W) = A\Gamma\left(\frac{1}{\alpha}+1\right)\frac{1}{Z(\mu t,\gamma)} + \left(\sum_{i=1}^{k}\left(\frac{A\Gamma\left(\frac{1}{\alpha}+1\right)}{a^{i-1}} + \frac{1}{Bb^{i-1}}\right)\left(\frac{1}{Z(\mu t,\gamma)}\sum_{j=1}^{k}\frac{\mu^j t^j e^{-j\psi}(j\psi)^{k-j}}{(j!)^{\gamma}(k-j)!}\right)\right)$$

Model 4

Let the test phase of the system X_1 follows Weibull distribution with parameter A, α, the fault removal time Y_1 follows Weibull distribution with parameter B, β and the number of faults detections N follows COM-Poisson Thomas process with parameters μ, γ and ψ.

where

$$E(X_1) = A\Gamma\left(\frac{1}{\alpha}+1\right), A > 0$$

$$E(Y_1) = \frac{1}{B}, B > 0$$

$$E[N(t)] = \frac{\mu(1+\psi)Z_{\mu}(\mu t,\gamma)}{Z(\mu t,\gamma)}$$

Then

$$E(W) = A\Gamma\left(\frac{1}{\alpha}+1\right)\frac{1}{Z(\mu t,\gamma)} + \left(\sum_{i=1}^{k}\left(\frac{A\Gamma\left(\frac{1}{\alpha}+1\right)}{a^{i-1}} + \frac{B\Gamma\left(\frac{1}{\beta}+1\right)}{b^{i-1}}\right) \times \left(\frac{1}{Z(\mu t,\gamma)}\sum_{j=1}^{k}\frac{\mu^j t^j e^{-j\psi}(j\psi)^{k-j}}{(j!)^{\gamma}(k-j)!}\right)\right)$$

Expected Total Cost

Cost Model 1

Let the test phase time X_1 follows exponential distribution with parameter A. Then

$$E(X_1) = \frac{1}{A}, A > 0$$

$$E_1(TC) = C_1\sum_{i=1}^{k}\frac{1}{Aa^{i-1}}\left(\frac{1}{Z(\mu t,\gamma)}\sum_{j=1}^{k}\frac{\mu^j t^j e^{-j\psi}(j\psi)^{k-j}}{(j!)^{\gamma}(k-j)!}\right)$$

Let the fault removal time Y_1 follows exponential distribution with parameter B. Then

$$E(Y_1) = \frac{1}{B}, B > 0$$

$$E_2(TC) = C_2\sum_{i=1}^{k}\frac{1}{Bb^{i-1}}\left(\frac{1}{Z(\mu t,\gamma)}\frac{1}{Z(\mu t,\gamma)}\sum_{j=1}^{k}\frac{\mu^j t^j e^{-j\psi}(j\psi)^{k-j}}{(j!)^{\gamma}(k-j)!}\right)$$

Then the expected total cost is,

$$E(TC) = C_1\sum_{i=1}^{k}\frac{1}{Aa^{i-1}}\left(\frac{1}{Z(\mu t,\gamma)}\frac{1}{Z(\mu t,\gamma)}\sum_{j=1}^{k}\frac{\mu^j t^j e^{-j\psi}(j\psi)^{k-j}}{(j!)^{\gamma}(k-j)!}\right) + C_2\sum_{i=1}^{k}\frac{1}{Bb^{i-1}}\left(\frac{1}{Z(\mu t,\gamma)}\sum_{j=1}^{k}\frac{\mu^j t^j e^{-j\psi}(j\psi)^{k-j}}{(j!)^{\gamma}(k-j)!}\right) + C_3(1+ct)e^{-ct}M$$

Cost Model 2

Let the test phase time X_1 follows exponential distribution with parameter A. Then

$$E(X_1) = \frac{1}{A}, A > 0$$

$$E_1(TC) = C_1 \sum_{i=1}^{k} \frac{1}{Aa^{i-1}} \cdot \frac{1}{Z(\mu t, \gamma)} \sum_{j=1}^{k} \frac{\mu^j t^j e^{-j\psi} (j\psi)^{k-j}}{(j!)^{\gamma}(k-j)!}.$$

Let the fault removal time Y_1 follows Weibull distribution with parameter B, β. Then

$$E(Y_1) = B\Gamma\left(\frac{1}{\beta}+1\right), B > 0$$

$$E_2(TC) = C_2 \sum_{i=1}^{k} \left(\frac{B\Gamma\left(\frac{1}{\beta}+1\right)}{b^{i-1}}\right) \cdot \frac{1}{Z(\mu t, \gamma)} \sum_{j=1}^{k} \frac{\mu^j t^j e^{-j\psi} (j\psi)^{k-j}}{(j!)^{\gamma}(k-j)!}$$

Then the expected total cost is,

$$E(TC) = C_1 \sum_{i=1}^{k} \frac{1}{Aa^{i-1}} \cdot \frac{1}{Z(\mu t, \gamma)} \sum_{j=1}^{k} \frac{\mu^j t^j e^{-j\psi} (j\psi)^{k-j}}{(j!)^{\gamma}(k-j)!} + C_2 \sum_{i=1}^{k} \left(\frac{B\Gamma\left(\frac{1}{\beta}+1\right)}{b^{i-1}}\right) \cdot \frac{1}{Z(\mu t, \gamma)} \sum_{j=1}^{k} \frac{\mu^j t^j e^{-j\psi} (j\psi)^{k-j}}{(j!)^{\gamma}(k-j)!} + C_3(1+ct)e^{-ct}M$$

Cost Model 3

Let the test phase time X_1 follows exponential distribution with parameter A, α. Then

$$E(X_1) = A\Gamma\left(\frac{1}{\alpha}+1\right), A > 0;$$

$$E_1(TC) = C_1 \sum_{i=1}^{k} \left(\frac{A\Gamma\left(\frac{1}{\alpha}+1\right)}{a^{i-1}}\right) \cdot \frac{1}{Z(\mu t, \gamma)} \sum_{j=1}^{k} \frac{\mu^j t^j e^{-j\psi} (j\psi)^{k-j}}{(j!)^{\gamma}(k-j)!}$$

Let the fault removal time Y_1 follows exponential distribution with parameter B. Then

$$E(Y_1) = \frac{1}{B}, B > 0$$

$$E_2(TC) = C_2 \sum_{i=1}^{k} \frac{1}{Bb^{i-1}} \cdot \frac{1}{Z(\mu t, \gamma)} \sum_{j=1}^{k} \frac{\mu^j t^j e^{-j\psi} (j\psi)^{k-j}}{(j!)^{\gamma}(k-j)!}.$$

Then the expected total cost is,

$$E(TC) = C_1 \sum_{i=1}^{k} \left(\frac{A\Gamma\left(\frac{1}{\alpha}+1\right)}{a^{i-1}}\right) \cdot \frac{1}{Z(\mu t, \gamma)} \sum_{j=1}^{k} \frac{\mu^j t^j e^{-j\psi} (j\psi)^{k-j}}{(j!)^{\gamma}(k-j)!} + C_2 \sum_{i=1}^{k} \frac{1}{Bb^{i-1}} \cdot \frac{1}{Z(\mu t, \gamma)} \sum_{j=1}^{k} \frac{\mu^j t^j e^{-j\psi} (j\psi)^{k-j}}{(j!)^{\gamma}(k-j)!} + C_3(1+ct)e^{-ct}M$$

Cost Model 4

Let the test phase time X_1 follows Weibull distribution with parameter A, α. Then

$$E(X_1) = A\Gamma\left(\frac{1}{\alpha}+1\right), A > 0;$$

$$E_1(TC) = C_1 \sum_{i=1}^{k} \left(\frac{A\Gamma\left(\frac{1}{\alpha}+1\right)}{a^{i-1}}\right)\left(\frac{1}{Z(\mu t, \gamma)}\sum_{j=1}^{k}\frac{\mu^j t^j e^{-j\psi}(j\psi)^{k-j}}{(j!)^{\gamma}(k-j)!}\right)$$

Let the fault removal time Y_1 follows Weibull distribution with parameter B, β. Then

$$E(Y_1) = B\Gamma\left(\frac{1}{\beta}+1\right), B > 0$$

$$E_2(TC) = C_2 \sum_{i=1}^{k} \left(\frac{B\Gamma\left(\frac{1}{\beta}+1\right)}{b^{i-1}}\right)\left(\frac{1}{Z(\mu t, \gamma)}\sum_{j=1}^{k}\frac{\mu^j t^j e^{-j\psi}(j\psi)^{k-j}}{(j!)^{\gamma}(k-j)!}\right)$$

Then the expected total cost is,

$$E(TC) = C_1 \sum_{i=1}^{k} \left(\frac{A\Gamma\left(\frac{1}{\alpha}+1\right)}{a^{i-1}}\right)\left(\frac{1}{Z(\mu t, \gamma)}\sum_{j=1}^{k}\frac{\mu^j t^j e^{-j\psi}(j\psi)^{k-j}}{(j!)^{\gamma}(k-j)!}\right)$$
$$+ C_2 \sum_{i=1}^{k} \left(\frac{B\Gamma\left(\frac{1}{\beta}+1\right)}{b^{i-1}}\right)\left(\frac{1}{Z(\mu t, \gamma)}\sum_{j=1}^{k}\frac{\mu^j t^j e^{-j\psi}(j\psi)^{k-j}}{(j!)^{\gamma}(k-j)!}\right)$$
$$+ C_3(1 + ct)e^{-ct}M$$

4 Conclusion

In this paper, COM-Poisson Thomas process is studied and its properties are discussed. Traffic accidents and fatalities model, shock model, natural disaster model, game model and system reliability models are framed.

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